Internet Appendix for Semiparametric Conditional Quantile Models for Financial Returns and Realized Volatility

Filip Žikeš* Jozef Baruník†

September 6, 2014

In this appendix, we study the implications of the measurement error induced by replacing the unobserved volatility components by their sample counterparts in our linear quantile regression models and provide sufficient conditions ensuring that the measurement error vanishes asymptotically. We also report the results of the out-of-sample exercise for the financial crisis of 2008-9 discussed in Section 7.3 of the paper.

1 Measurement error problem

The quantile regression models proposed in the paper are based on realized measures rather than the true, unobserved components of price variation. Asymptotic theory for the realized measures dictates that as the number of intraday observations grows without bound, the realized measures approach their unobserved counterparts and, equivalently, the measurement error associated with the realized measures approaches zero. Thus, under certain conditions it may be feasible to obtain, asymptotically, conditional quantiles for the true quadratic variation or any of its components. Whether or not this is desirable depends on the application at hand. If, for example, the objective is to estimate value-at-risk for variance swap positions, one need not worry about the measurement error problem, since here the goal is to estimate the quantiles of the realized volatility calculated at a fixed sampling frequency stipulated by the variance swap contract, i.e. $q_\alpha(RV_{t+1,M}|\Omega_t)$ for some fixed M.

*Corresponding author: Bank of England, Macro Financial Analysis Division, Threadneedle Street, London EC2R 8AH. Phone: +44 20 7601 5092. Email: filip.zikes@bankofengland.co.uk.

†Institute of Economic Studies, Charles University, Opletalova 21, 110 00, Prague, CR and Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Pod Vodarenskou Vezi 4, 182 00, Prague, Czech Republic.

However, if the goal is to estimate the quantiles of future asset returns volatility, one needs to make sure that the impact of the measurement error vanishes so that one indeed obtains quantiles for the true quadratic variation, $q_\alpha(QV_{t+1}|\Omega_t)$, rather than realized volatility.

In this section, we provide sufficient conditions ensuring that the feasible objective function, $QR_{T,M}$, based on the realized measures converges in probability to the infeasible one, QR_T, based on the true unobserved components of quadratic variation, uniformly on the parameter space. If these conditions are satisfied we obtain, asymptotically, the desired quantiles of the quadratic variation, $q_\alpha(QV_{t+1}|\Omega_t)$, rather than the realized variance, $q_\alpha(RV_{t+1}|\Omega_t)$. The sufficient conditions depend on the properties of the measurement errors associated with the realized measures, which in turn depend on the behavior of the volatility and jump processes driving the logarithmic price, and on the relative rate of growth of M and T.

To establish the asymptotic equivalence, we follow the double-asymptotic approach of Corradi, Distaso & Swanson (2011), who study fully nonparametric estimators of conditional distributions of integrated variance using realized measures. In doing so, they establish some useful results regarding the rate of decay of moments of the measurement error associated with a number of realized measures. We extend these results to the case of realized volatility and median realized volatility in the presence of jumps and employ these to prove the asymptotic negligibility of the measurement error for the estimation of conditional quantiles.

We will need the following assumptions:

(A1) The logarithmic price process follows (1) with $\mu_t \equiv 0$,

(A2) The volatility process $\{\sigma_t\}$ is a strong mixing with size $-2r/(r-2), r > 2$ satisfying $\mathbb{E}[(\sigma_t^2)^{2(k+r)}] < \infty$, and the jump sizes satisfy $\mathbb{E}[\kappa^{2k}] < \infty$ for some $k \geq 2$.

(A3) The counting process L_t is a Poisson process with strictly stationary intensity.

Assumption A1 specifies the data generating process. To simplify the proofs we assume that the drift is equal to zero. Assumptions A2 and A3 ensure that the moments of the measurement errors associated with $IV_{t,M}$ and $JV_{t,M}$ exist and decay sufficiently fast, as the following Lemma shows.

Lemma 1 *Let $N_{t,M}^{(c)} := IV_t - IV_{t,M}$ and $N_{t,M}^{(d)} := JV_t - JV_{t,M}$. Then under assumptions A1-A2, $\mathbb{E}[|N_{t,M}^{(c)}|^k] = O(M^{-k/2})$. If, in addition, A3 holds then $\mathbb{E}[|N_{t,M}^{(d)}|^k] = O(M^{-k/2})$.*

Proof The first part of the result is proved by Corradi et al. (2011) for bi-power and tri-power variation of Barndorff-Nielsen & Shephard (2004). Using the same line of argument as in Corradi et al. (2011), one can show that the same result holds for the median realized volatility as well and we omit the proof to save space.

To prove the second result, write

$$\begin{aligned}
|JV_{t,M} - JV_t| &= |RV_{t,M} - IV_{t,M} - JV_t|, \\
&\le |RV_{t,M} - IV_t - JV_t| + |IV_{t,M} - IV_t|, \\
&\equiv A_{t,M} + B_{t,M}.
\end{aligned}$$

$B_{t,M}$ was discussed above so we need to focus on $A_{t,M}$.

$$\begin{aligned}
A_{t,M} &= \left| \sum_{i=1}^{M} \left(\int_{t_{i-1}}^{t_i} \sigma_u \mathrm{d}W_u + \sum_{l=1}^{\Delta_i L_t} \kappa_l \right)^2 - \int_{t-1}^{t} \sigma_u^2 \mathrm{d}u - \sum_{l=1}^{\Delta L_t} \kappa_l^2 \right|, \\
&\le \left| \sum_{i=1}^{M} \left(\int_{t_{i-1}}^{t_i} \sigma_u \mathrm{d}W_u \right)^2 - \int_{t_{i-1}}^{t_i} \sigma_u^2 \mathrm{d}u \right| + \left| 2 \sum_{i=1}^{M} \left(\int_{t_{i-1}}^{t_i} \sigma_u \mathrm{d}W_u \right) \left(\sum_{l=1}^{\Delta_i L_t} \kappa_l \right) \right| \\
&+ \left| \sum_{i=1}^{M} \left[\left(\sum_{l=1}^{\Delta_i L_t} \kappa_l \right)^2 - \sum_{l=1}^{\Delta_i L_t} \kappa_l^2 \right] \right| \\
&\equiv C_{t,M} + D_{t,M} + E_{t,M}.
\end{aligned}$$

Now $C_{t,M}$ is the measurement error associated with realized volatility in the absence of jumps and by Corradi et al. (2011) we have $\mathbb{E}(|C_{t,M}|^k) = O(M^{-k/2})$. Given Assumptions A2 (existence of moments of jumps) and A3 (finite-activity), we can proceed by assuming that there is at most one jump in every time interval $[t_{i-1}, t_i]$. Then $E_{t,M} = 0$ and write $D_{t,M}$ as

$$\begin{aligned}
D_{t,M} &= 2 \sum_{i=1}^{M} \left| \int_{t_{i-1}}^{t_i} \sigma_u \mathrm{d}W_u \right| |\kappa_{t_i}| \mathbf{1}_{\{\Delta_i L_t = 1\}} \\
&\le 2 \sum_{i=1}^{M} \left| \int_{t_{i-1}}^{t_i} \sigma_{t_{i-1}} \mathrm{d}W_u \right| |\kappa_{t_i}| \mathbf{1}_{\{\Delta_i L_t = 1\}} + 2 \sum_{i=1}^{M} \left| \int_{t_{i-1}}^{t_i} (\sigma_u - \sigma_{t_{i-1}}) \mathrm{d}W_u \right| |\kappa_{t_i}| \mathbf{1}_{\{\Delta_i L_t = 1\}} \\
&= D_{t,M}^{(1)} + D_{t,M}^{(2)}.
\end{aligned}$$

For simplicity we focus on the case of $k = 2$ noting that the case of $k > 2$ can be treated analogously. Taking expectations,

$$\begin{aligned}\mathbb{E}[|D_{t,M}^{(1)}|^2] &= 4\sum_{i_1=1}^{M}\sum_{i_2=1}^{M}\mathbb{E}\left[\sigma_{t_{i_1-1}}\sigma_{t_{i_2-1}}\left|\int_{t_{i_1-1}}^{t_{i_1}}\mathrm{d}W_u\right|\left|\int_{t_{i_2-1}}^{t_{i_2}}\mathrm{d}W_u\right||\kappa_{t_{i_1}}||\kappa_{t_{i_2}}|\mathbf{1}_{\{\Delta_{i_1}L_t=1\}}\mathbf{1}_{\{\Delta_{i_2}L_t=1\}}\right]\\ &= 4\sum_{i_1=1}^{M}\sum_{i_2=1}^{M}\mathbb{E}\left[\sigma_{t_{i_1-1}}\sigma_{t_{i_2-1}}\left|\int_{t_{i_1-1}}^{t_{i_1}}\mathrm{d}W_u\right|\left|\int_{t_{i_2-1}}^{t_{i_2}}\mathrm{d}W_u\right|\right]\\ &\qquad\qquad\times\mathbb{E}[|\kappa_{t_{i_1}}||\kappa_{t_{i_2}}|]\mathbb{E}[\mathbf{1}_{\{\Delta_{i_1}L_t=1\}}\mathbf{1}_{\{\Delta_{i_2}L_t=1\}}]\end{aligned}$$

By Hölder inequality, the first expectation is $O(M^{-1})$ provided that $\mathbb{E}[\sigma_u^2] < \infty$, while the second expectation is $O(M^{-2})$ if $i_1 \neq i_2$ and $O(M^{-1})$ if $i_1 = i_2$, provided that $\mathbb{E}[\kappa_j^2] < \infty$. Thus, $\mathbb{E}[|D_{t,M}^{(1)}|^2] = O(M^{-1})$. Finally, $\mathbb{E}[|D_{t,M}^{(2)}|^2]$ can not be of higher order than $\mathbb{E}[|D_{t,M}^{(1)}|^2]$, see Corradi et al. (2011) for details. □

The first result of the Lemma is the same as in Lemma 1 of Corradi et al. (2011), who prove this for a number of different realized measures of integrated variance. Assuming, in addition, A3 allows us to establish similar result for the measure of jump variation based on the difference between realized variance and median realized variance. Given Lemma 1, we then have the following:

Proposition 1 *Under assumptions (A1) - (A3), if* $T^{\frac{2}{2k-1}}M^{-1/2} \to 0$ *as* $T, M \to \infty$ *and* Θ *is a compact parameter space, then* $\sup_{\boldsymbol{\beta}\in\Theta}|QR_{T,M}(\boldsymbol{\beta}) - QR_T(\boldsymbol{\beta})| \xrightarrow{p} 0$.

Proof To save space, we prove the proposition for $V_t = \{IV_{t,M}, JV_{t,M}\}$ and $\beta_Z = \mathbf{0}$, noting that the others cases can be treated analogously. Simplifying notation we will write $\boldsymbol{\beta} = \boldsymbol{\beta}(\alpha)$ since α is fixed throughout. Define

$$\begin{aligned} z_{t+1}(\boldsymbol{\beta}) &= QV_{t+1} - \beta_0 - \beta_1 IV_t - \beta_2 JV_t,\\ z_{t+1,M}(\boldsymbol{\beta}) &= RV_{t+1,M} - \beta_0 - \beta_1 IV_{t,M} - \beta_2 JV_{t,M},\end{aligned}$$

and write

$$
\begin{aligned}
& |QR_{T,M}(\boldsymbol{\beta}) - QR_T(\boldsymbol{\beta})| \\
= & \left| \frac{1}{T}\sum_{t=0}^{T-1} z_{t+1,M}(\boldsymbol{\beta})[\alpha - \mathbf{1}\{z_{t+1,M}(\boldsymbol{\beta}) \leq 0\}] - \frac{1}{T}\sum_{t=0}^{T-1} z_{t+1}(\boldsymbol{\beta})[\alpha - \mathbf{1}\{z_{t+1}(\boldsymbol{\beta}) \leq 0\}] \right| \\
= & \left| \frac{1}{T}\sum_{t=0}^{T-1} (z_{t+1,M}(\boldsymbol{\beta}) - z_{t+1}(\boldsymbol{\beta}))(\alpha - \mathbf{1}\{z_{t+1,M}(\boldsymbol{\beta}) \leq 0\}) - (\mathbf{1}\{z_{t+1,M}(\boldsymbol{\beta}) \leq 0\} - \mathbf{1}\{z_{t+1}(\boldsymbol{\beta}) \leq 0\}) z_{t+1}(\boldsymbol{\beta}) \right| \\
\leq & \frac{1}{T}\sum_{t=0}^{T-1} |z_{t+1,M}(\boldsymbol{\beta}) - z_{t+1}(\boldsymbol{\beta})| + \frac{1}{T}\sum_{t=0}^{T-1} |z_{t+1}(\boldsymbol{\beta})||\mathbf{1}\{z_{t+1,M}(\boldsymbol{\beta}) \leq 0\} - \mathbf{1}\{z_{t+1}(\beta) \leq 0\}| \\
\equiv & A_{T,M} + B_{T,M}.
\end{aligned}
$$

Now

$$
A_{T,M} = \frac{1}{T}\sum_{t=0}^{T-1} |N_{t,M} - \beta_1 N_{t,M}^{(c)} - \beta_2 N_{t,M}^{(d)}| \leq \frac{1}{T}\sum_{t=1}^{T} |N_{t,M}| + \bar{\beta}_1 |N_{t,M}^{(c)}| + \bar{\beta}_2 |N_{t,M}^{(d)}|,
$$

where $N_{t,M} = RV_{t,M} - IV_t - JV_t$, $\bar{\beta}_1 = \max|\beta_1|$ and $\bar{\beta}_2 = \max|\beta_2|$, which are well-defined since Θ is compact. It follows by Markov inequality, stationarity and Lemma 1 that $A_{T,M} = O_p(M^{-1/2})$ uniformly in $\boldsymbol{\beta}$.

Turning to $B_{t,M}$, we have

$$
\begin{aligned}
B_{T,M} &= \frac{1}{T}\sum_{t=0}^{T-1} |z_{t+1}(\boldsymbol{\beta})||\mathbf{1}\{z_{t+1}(\boldsymbol{\beta}) + N_{t,M} - \beta_1 N_{t,M}^{(c)} - \beta_2 N_{t,M}^{(d)} \leq 0\} - \mathbf{1}\{z_{t+1}(\boldsymbol{\beta}) \leq 0\}| \\
&\leq \frac{1}{T}\sum_{t=0}^{T-1} |z_{t+1}(\boldsymbol{\beta})| \mathbf{1}\{-\sup_t |N_{t,M}| - \bar{\beta}_1 \sup_t |N_{t,M}^{(c)}| - \bar{\beta}_2 \sup_t |N_{t,M}^{(d)}| \leq z_{t+1}(\boldsymbol{\beta})\} \\
&\qquad \times \mathbf{1}\{z_{t+1}(\boldsymbol{\beta}) \leq \sup_t |N_{t,M}| + \bar{\beta}_1 \sup_t |N_{t,M}^{(c)}| + \bar{\beta}_2 \sup_t |N_{t,M}^{(d)}|\}.
\end{aligned}
$$

By Lemma 1, $\mathbb{E}|N_{t,M}^{(d)}| = O(M^{-k/2})$, and hence by Markov inequality

$$
\begin{aligned}
\mathbb{P}\left[\sup_t T^{-\frac{2}{2k-1}} M^{1/2} |N_{t,M}^{(d)}| > \epsilon\right] &\leq \sum_{t=1}^{T} \mathbb{P}\left[T^{-\frac{2}{2k-1}} M^{1/2} |N_{t,M}^{(d)}| > \epsilon\right], \\
&\leq \frac{1}{\epsilon} T^{1-\frac{2k}{2k-1}} M^{k/2} \mathbb{E}[|N_{t,M}^{(d)}|^k], \\
&= O(T^{1-\frac{2k}{2k-1}}).
\end{aligned}
$$

and similarly for $N_{t,M}^{(c)}$ and $N_{t,M}$. It follows that there exists a constant c such that with probability approaching one as $T, M \to \infty$

$$B_{T,M} \leq \frac{1}{T}\sum_{t=1}^{T}|z_t(\boldsymbol{\beta})|\mathbf{1}\{-bc\epsilon T^{\frac{2}{2k-1}}M^{-1/2} \leq z_{t+1}(\boldsymbol{\beta}) \leq bc\epsilon T^{\frac{2}{2k-1}}M^{-1/2}\}, \quad (1)$$

$$\equiv B'_{t,M}, \quad (2)$$

where $b = 1 + \bar{\beta}_1 + \bar{\beta}_2$. Thus, following Corradi et al. (2011) we can proceed by conditioning on a set on which (1) holds and focus on $B'_{t,M}$. By Markov and Hölder inequalities,

$$\mathbb{P}(B'_{T,M} > \eta) \leq \frac{1}{\eta}\frac{1}{T}\sum_{t=0}^{T-1}\mathbb{E}(z_t^2(\boldsymbol{\beta}))^{1/2}\mathbb{P}(-bc\epsilon T^{\frac{2}{2k-1}}M^{-1/2} \leq z_{t+1}(\boldsymbol{\beta}) \leq bc\epsilon T^{\frac{2}{2k-1}}M^{-1/2})^{1/2}.$$

Since Θ is compact, Assumption A2 implies that $\mathbb{E}(z_t^2(\boldsymbol{\beta}))^{1/2} < \infty$ and if $T^{\frac{2}{2k-1}}M^{-1/2} \to 0$ as $T, M \to \infty$, we have $\mathbb{P}(-bc\epsilon T^{\frac{2}{2k-1}}M^{-1/2} \leq z_{t+1}(\boldsymbol{\beta}) \leq bc\epsilon T^{\frac{2}{2k-1}}M^{-1/2})^{1/2} \to 0$, uniformly in $\boldsymbol{\beta}$. The statement in the proposition then follows. □

The proposition shows that the number of intraday observations (M) has to grow faster than a power of (T) for the contribution of the measurement error associated with the realized measures of integrated variance and jump variation to degenerate in the limit. How faster M must grow depends on the the number of moments the volatility and jump processes possess. If all moments exist (i.e. $k = \infty$), we obtain the intuitive result that the contribution of the measurement error is driven entirely by discretization (finite M), i.e. it suffices to have $M \to \infty$ regardless of how fast this happens relative to $T \to \infty$. The reason we cannot establish this intuitive result for any $k \geq 2$ is due to the fact that the standard mean-value argument does not apply due to the non-differentiability of the objective function. To circumvent this problem, we have to ensure that $\sup_t |IV_{t,M} - IV_t|$ and $\sup_t |JV_{t,M} - JV_t|$ decay sufficiently fast, and this in turn depends on k and the relative rate of growth of M and T.

2 Results for the out-of-sample exercise for the 2008-9 crisis

		S&P 500					WTI Crude Oil				
	α	0.05	0.10	0.50	0.90	0.95	0.05	0.10	0.50	0.90	0.95
ARFIMA	$\hat{\alpha}$	0.070	0.131	0.468	0.876	0.930	0.052	0.108	0.498	0.878	0.942
	DQ	6.089	10.646	12.038	8.572	5.775	4.931	1.709	6.512	5.187	8.286
	p-val	0.557	0.099	0.054	0.221	0.589	0.703	0.959	0.360	0.555	0.248
SAV	$\hat{\alpha}$	0.060	0.106	0.506	0.886	0.946	0.062	0.116	0.482	0.888	0.940
	DQ	5.820	4.228	6.797	4.164	2.347	4.669	5.469	7.599	4.329	4.706
	p-val	0.595	0.653	0.340	0.666	0.936	0.738	0.482	0.259	0.644	0.738
RSAV1	$\hat{\alpha}$	0.058	0.102	0.506	0.876	0.944	0.050	0.116	0.490	0.886	0.940
	DQ	1.527	4.605	6.818	3.732	5.514	7.117	5.198	12.161	6.100	5.145
	p-val	0.962	0.624	0.348	0.719	0.609	0.413	0.561	0.069	0.424	0.677
RSAV2	$\hat{\alpha}$	0.056	0.110	0.498	0.918	0.962	0.054	0.096	0.474	0.880	0.934
	DQ	7.322	5.294	8.833	11.894	8.815	5.240	0.684	7.642	6.330	7.211
	p-val	0.394	0.527	0.164	0.076	0.201	0.674	0.996	0.244	0.409	0.422
AS	$\hat{\alpha}$	0.074	0.120	0.486	0.847	0.914	0.062	0.116	0.506	0.878	0.924
	DQ	8.702	8.291	9.050	16.966	15.069	7.692	6.042	9.916	5.765	10.221
	p-val	0.224	0.253	0.162	0.014	0.018	0.342	0.451	0.125	0.441	0.134
RAS	$\hat{\alpha}$	0.050	0.112	0.484	0.882	0.944	0.046	0.096	0.454	0.886	0.942
	DQ	7.245	3.733	7.328	8.572	16.191	7.725	0.370	14.913	5.267	11.247
	p-val	0.411	0.757	0.272	0.254	0.004	0.347	1.000	0.021	0.561	0.084
LQR1	$\hat{\alpha}$	0.080	0.130	0.488	0.878	0.934	0.080	0.144	0.500	0.874	0.928
	DQ	13.168	7.155	11.315	5.951	12.210	14.980	10.733	4.314	5.127	7.879
	p-val	0.029	0.337	0.075	0.470	0.056	0.011	0.120	0.629	0.566	0.306
LQR2	$\hat{\alpha}$	0.068	0.120	0.496	0.910	0.950	0.050	0.104	0.480	0.904	0.950
	DQ	11.033	4.749	10.897	3.842	5.420	3.204	3.206	6.186	3.169	3.224
	p-val	0.097	0.585	0.089	0.713	0.653	0.901	0.790	0.388	0.814	0.886
LQR3	$\hat{\alpha}$	0.066	0.122	0.496	0.902	0.944	0.054	0.104	0.478	0.898	0.946
	DQ	5.666	4.911	14.020	4.152	2.477	4.937	2.689	12.404	4.397	2.355
	p-val	0.616	0.572	0.026	0.685	0.941	0.721	0.841	0.054	0.647	0.956

Table 1: Absolute out-of-sample performance of alternative conditional quantile models for daily S&P500 and WTI Crude Oil futures returns during the 2008-2009 period. One-step-ahead forecasts. For each model and quantile (α) we report the unconditional coverage ($\hat{\alpha}$), the Berkowitz et al. (2011) test statistic for correct dynamic specification (DQ) and the corresponding Monte Carlo-based p-value (p-val).

		h=1					h=5					h=10				
	α	0.05	0.10	0.50	0.90	0.95	0.05	0.10	0.50	0.90	0.95	0.05	0.10	0.50	0.90	0.95
ARFIMA	$\hat{\alpha}$	0.070	0.131	0.468	0.876	0.930	0.048	0.108	0.492	0.900	0.966	0.058	0.118	0.476	0.928	0.976
	$\hat{L}$	0.167	0.287	0.627	0.271	0.154	0.347	0.573	1.219	0.500	0.290	0.522	0.852	1.678	0.668	0.399
	DM	-2.412†	-3.379†	0.633	0.609	0.034	-1.681†	-1.571†	0.110	0.785	0.815	-0.297	-0.352	0.737	0.798	1.126
SAV	$\hat{\alpha}$	0.060	0.106	0.506	0.886	0.946	0.066	0.133	0.548	0.859	0.924	0.090	0.141	0.542	0.853	0.910
	$\hat{L}$	0.190	0.309	0.630	0.286	0.172	0.409	0.646	1.262	0.573	0.364	0.668	1.006	1.703	0.796	0.519
	DM	1.370*	0.728	0.960	2.773*	2.813*	2.474*	2.030*	1.526*	1.608*	2.309*	2.440*	2.874*	0.866	1.457*	1.789*
RSAV1	$\hat{\alpha}$	0.058	0.102	0.506	0.876	0.944	0.058	0.116	0.512	0.855	0.926	0.084	0.139	0.516	0.831	0.906
	$\hat{L}$	0.175	0.293	0.637	0.273	0.161	0.393	0.620	1.250	0.549	0.320	0.578	0.960	1.754	0.761	0.464
	DM	-0.502	-1.366†	2.201*	1.583*	2.159*	1.964*	1.175	0.879	1.840*	1.745*	1.310*	1.880*	1.841*	1.566*	1.492*
RSAV2	$\hat{\alpha}$	0.056	0.110	0.498	0.918	0.962	0.056	0.100	0.514	0.882	0.938	0.090	0.133	0.480	0.837	0.902
	$\hat{L}$	0.173	0.295	0.631	0.272	0.155	0.407	0.626	1.285	0.553	0.315	0.737	1.079	1.861	0.785	0.480
	DM	-0.820	-1.039	1.144	1.417*	0.191	1.486*	1.390*	2.324*	1.671*	1.570*	1.423*	1.740*	3.096*	1.783*	1.951*
AS	$\hat{\alpha}$	0.074	0.120	0.486	0.847	0.914	0.076	0.141	0.494	0.795	0.855	0.124	0.169	0.538	0.789	0.839
	$\hat{L}$	0.188	0.310	0.629	0.278	0.165	0.407	0.644	1.265	0.561	0.352	0.692	1.019	1.754	0.830	0.568
	DM	1.165	1.035	0.764	1.334*	1.452*	2.176*	2.144*	1.668*	1.577*	2.019*	2.392*	2.784*	1.656*	1.427*	1.704*
RAS1	$\hat{\alpha}$	0.050	0.112	0.484	0.882	0.944	0.052	0.114	0.476	0.821	0.900	0.090	0.135	0.444	0.807	0.851
	$\hat{L}$	0.181	0.297	0.634	0.275	0.165	0.429	0.651	1.269	0.545	0.324	0.728	1.109	1.934	0.852	0.591
	DM	0.460	-0.603	1.305*	1.482*	2.095*	1.867*	1.643*	1.333*	1.659*	1.650*	1.686*	1.924*	2.896*	1.342*	1.448*
LQR1	$\hat{\alpha}$	0.080	0.130	0.488	0.878	0.934	0.060	0.110	0.514	0.862	0.928	0.092	0.144	0.498	0.864	0.936
	$\hat{L}$	0.175	0.299	0.625	0.273	0.159	0.364	0.587	1.223	0.522	0.305	0.567	0.889	1.669	0.686	0.397
	DM	-0.687	-0.636	0.569	1.534*	1.044	-0.030	-0.370	1.158	1.674*	1.379*	1.414*	0.691	1.166	1.340*	1.045
LQR2	$\hat{\alpha}$	0.068	0.120	0.496	0.910	0.950	0.048	0.100	0.520	0.900	0.946	0.074	0.140	0.510	0.904	0.956
	$\hat{L}$	0.178	0.302	0.625	0.266	0.154	0.365	0.593	1.218	0.495	0.284	0.532	0.864	1.657	0.652	0.381
LQR3	$\hat{\alpha}$	0.066	0.122	0.496	0.902	0.944	0.050	0.098	0.516	0.900	0.948	0.072	0.138	0.512	0.906	0.962
	$\hat{L}$	0.180	0.302	0.626	0.267	0.153	0.364	0.591	1.220	0.488	0.279	0.532	0.866	1.656	0.646	0.379
	DM	0.947	0.024	0.444	0.568	-0.762	-0.155	-0.574	0.507	-2.137†	-1.259	-0.021	0.779	-0.140	-1.765†	-0.540

Table 2: Relative performance of alternative out-of-sample forecasts of S&P 500 futures return quantiles. For each model, quantile (α) and forecasts horizon (h), we report the unconditional coverage ($\hat{\alpha}$), the value of the tick-loss function ($\hat{L}$) and the Diebold-Mariano test statistic for equal predictive accuracy with the linear quantile regression model LQR2 serving as the benchmark. We use $*$ to denote significantly less accurate forecasts and †to denote significantly more accurate forecasts with respect to the benchmark at the 5% significance level.

		h=1					h=5					h=10				
	α	0.05	0.10	0.50	0.90	0.95	0.05	0.10	0.50	0.90	0.95	0.05	0.10	0.50	0.90	0.95
ARFIMA	$\hat{\alpha}$	0.052	0.108	0.498	0.878	0.942	0.082	0.127	0.504	0.892	0.958	0.096	0.151	0.486	0.886	0.960
	$\hat{L}$	0.238	0.418	0.968	0.429	0.253	0.692	1.106	2.309	0.956	0.580	1.081	1.736	3.400	1.218	0.669
	DM	-3.388†	-2.372†	-0.485	-2.728†	-2.338†	0.320	0.234	0.106	-0.765	-0.408	0.448	0.252	-0.032	-0.514	-0.732
SAV	$\hat{\alpha}$	0.062	0.116	0.482	0.888	0.940	0.090	0.149	0.504	0.884	0.920	0.104	0.171	0.528	0.876	0.912
	$\hat{L}$	0.266	0.447	0.988	0.450	0.278	0.883	1.283	2.307	1.026	0.666	1.288	2.062	3.650	1.329	0.812
	DM	0.873	0.979	1.437*	0.268	1.112	2.377*	2.107*	0.096	0.795	1.364*	2.098*	2.177*	1.430*	1.332*	1.373*
RSAV1	$\hat{\alpha}$	0.050	0.116	0.490	0.886	0.940	0.074	0.120	0.470	0.884	0.938	0.102	0.169	0.468	0.867	0.920
	$\hat{L}$	0.242	0.421	0.985	0.441	0.266	0.673	1.120	2.400	1.021	0.652	1.248	2.070	3.510	1.427	0.828
	DM	-1.803†	-1.953†	0.972	-1.333†	-0.329	0.055	0.787	2.635*	0.578	1.387*	2.220*	2.422*	1.181	1.805*	1.122
RSAV2	$\hat{\alpha}$	0.054	0.096	0.474	0.880	0.934	0.054	0.131	0.357	0.871	0.932	0.068	0.088	0.321	0.797	0.873
	$\hat{L}$	0.243	0.421	0.967	0.459	0.265	0.675	1.153	2.526	1.078	0.663	1.187	1.815	3.941	1.647	1.016
	DM	-1.631†	-2.255†	-0.673	1.219	-0.374	0.122	1.399*	1.917*	1.318*	2.294*	2.397*	1.133	1.457*	1.562*	1.652*
AS	$\hat{\alpha}$	0.062	0.116	0.506	0.878	0.924	0.096	0.161	0.532	0.882	0.904	0.131	0.189	0.544	0.882	0.910
	$\hat{L}$	0.264	0.439	0.991	0.461	0.291	0.817	1.223	2.323	1.029	0.654	1.191	1.854	3.541	1.386	0.861
	DM	0.789	0.348	1.525*	1.236	2.694*	1.810*	1.572*	0.422	0.826	1.421*	1.252	1.102	0.649	1.262	1.223
RAS1	$\hat{\alpha}$	0.046	0.096	0.454	0.886	0.942	0.046	0.104	0.345	0.867	0.924	0.058	0.078	0.323	0.803	0.859
	$\hat{L}$	0.247	0.427	0.986	0.469	0.267	0.673	1.138	2.463	1.078	0.653	1.161	1.838	4.001	1.692	1.119
	DM	-1.127	-1.294†	1.187	2.029*	-0.135	0.045	1.553*	1.311*	1.278	1.808*	1.874*	1.155	1.537*	1.610*	1.753*
LQR1	$\hat{\alpha}$	0.080	0.144	0.500	0.874	0.928	0.084	0.146	0.518	0.868	0.922	0.104	0.164	0.510	0.866	0.930
	$\hat{L}$	0.269	0.458	0.967	0.449	0.269	0.733	1.135	2.296	1.024	0.610	1.110	1.827	3.414	1.282	0.729
	DM	1.248	1.733*	-0.657	0.189	0.193	1.007	0.693	-0.052	0.997	1.102	1.179	0.955	0.090	0.389	0.691
LQR2	$\hat{\alpha}$	0.050	0.104	0.480	0.904	0.950	0.050	0.100	0.482	0.890	0.944	0.072	0.118	0.484	0.894	0.946
	$\hat{L}$	0.258	0.436	0.974	0.448	0.268	0.670	1.087	2.298	1.000	0.595	0.999	1.681	3.399	1.269	0.710
LQR3	$\hat{\alpha}$	0.054	0.104	0.478	0.898	0.946	0.050	0.104	0.484	0.888	0.942	0.070	0.118	0.474	0.892	0.946
	$\hat{L}$	0.257	0.434	0.975	0.448	0.267	0.670	1.089	2.299	1.001	0.588	0.999	1.675	3.381	1.267	0.701
	DM	-0.400	-0.752	0.261	-0.036	-0.036	0.016	0.236	0.173	0.494	-0.715	-0.019	-0.526	-1.282†	-0.301	-1.342†

Table 3: Relative performance of alternative out-of-sample forecasts of WTI Crude Oil futures return quantiles. For each model, quantile (α) and forecasts horizon (h), we report the unconditional coverage ($\hat{\alpha}$), the value of the tick-loss function ($\hat{L}$) and the Diebold-Mariano test statistic for equal predictive accuracy with the linear quantile regression model LQR2 serving as the benchmark. We use $*$ to denote significantly less accurate forecasts and †to denote significantly more accurate forecasts with respect to the benchmark at the 5% significance level.

		in-sample				out-of-sample			
	α	0.5	0.75	0.90	0.95	0.5	0.75	0.90	0.95
ARFIMA	$\hat{\alpha}$	0.498	0.878	0.966	0.980	0.444	0.791	0.938	0.976
	DQ	74.181	104.183	52.361	21.797	35.835	17.893	15.770	11.786
	p-val	0.000	0.000	0.000	0.001	0.000	0.004	0.029	0.069
HARQ1	$\hat{\alpha}$	0.436	0.746	0.922	0.956	0.452	0.712	0.884	0.952
	DQ	18.870	4.995	9.493	3.025	9.456	4.255	10.526	10.834
	p-val	0.003	0.532	0.189	0.912	0.164	0.636	0.133	0.089
HARQ2	$\hat{\alpha}$	0.520	0.784	0.928	0.958	0.560	0.798	0.918	0.962
	DQ	27.833	12.917	18.581	9.330	12.252	9.072	3.571	6.244
	p-val	0.000	0.038	0.006	0.188	0.070	0.166	0.751	0.534
HARQ3	$\hat{\alpha}$	0.526	0.798	0.930	0.964	0.560	0.794	0.922	0.960
	DQ	39.639	18.071	22.131	7.511	10.015	7.239	4.453	5.689
	p-val	0.000	0.007	0.002	0.377	0.133	0.301	0.637	0.597

Table 4: Absolute out-of-sample performance of alternative conditional quantile models for daily S&P500 and WTI Crude Oil futures realized volatility. One-step-ahead forecasts. For each model and quantile (α) we report the unconditional coverage ($\hat{\alpha}$), the Berkowitz et al. (2011) test statistic for correct dynamic specification (DQ) and the corresponding Monte Carlo-based p-value (p-val).

		h=1				h=5				h=10			
	α	0.5	0.75	0.90	0.95	0.5	0.75	0.90	0.95	0.5	0.75	0.90	0.95
A. S&P 500													
ARFIMA	$\hat{\alpha}$	0.498	0.878	0.966	0.980	0.560	0.761	0.855	0.896	0.560	0.715	0.803	0.843
	$\hat{L}$	0.091	0.088	0.059	0.039	0.132	0.135	0.099	0.070	0.159	0.168	0.133	0.103
	DM	-10.719†	-7.313†	-5.219†	-4.422†	0.150	1.066	1.349*	1.308*	1.186	1.183	1.305*	1.361
HARQ1	$\hat{\alpha}$	0.436	0.746	0.922	0.956	0.456	0.734	0.890	0.940	0.448	0.724	0.856	0.908
	$\hat{L}$	0.135	0.125	0.082	0.055	0.135	0.125	0.082	0.053	0.148	0.145	0.098	0.067
	DM	1.594*	1.600*	0.253	-0.518	1.028	-0.075	0.326	-0.022	0.327	-0.241	-0.170	-0.160
HARQ2	$\hat{\alpha}$	0.520	0.784	0.928	0.958	0.520	0.756	0.898	0.946	0.520	0.754	0.876	0.916
	$\hat{L}$	0.131	0.120	0.081	0.056	0.131	0.125	0.081	0.053	0.147	0.146	0.099	0.067
HARQ3	$\hat{\alpha}$	0.526	0.798	0.930	0.964	0.528	0.764	0.894	0.934	0.542	0.756	0.866	0.920
	$\hat{L}$	0.131	0.123	0.083	0.057	0.133	0.126	0.084	0.054	0.147	0.148	0.100	0.068
	DM	0.311	1.483*	1.069	0.853	2.160*	1.038	1.631*	1.072	0.862	3.711*	2.782*	0.776
B: WTI Crude Oil													
ARFIMA	$\hat{\alpha}$	0.444	0.791	0.938	0.976	0.460	0.699	0.823	0.890	0.464	0.663	0.757	0.815
	$\hat{L}$	0.155	0.129	0.078	0.050	0.147	0.135	0.083	0.050	0.169	0.165	0.110	0.073
	DM	-7.409†	-9.934†	-7.526†	-5.331†	1.160	1.336*	0.908	0.550	1.450*	1.713*	1.564*	1.557
HARQ1	$\hat{\alpha}$	0.452	0.712	0.884	0.952	0.458	0.706	0.882	0.946	0.444	0.696	0.862	0.922
	$\hat{L}$	0.193	0.172	0.105	0.062	0.134	0.119	0.069	0.042	0.136	0.119	0.075	0.044
	DM	1.004	0.285	0.028	-1.264	0.191	0.235	-0.686	-1.145	0.259	-0.130	0.147	0.026
HARQ2	$\hat{\alpha}$	0.560	0.798	0.918	0.962	0.610	0.826	0.932	0.962	0.616	0.816	0.912	0.950
	$\hat{L}$	0.190	0.171	0.105	0.066	0.133	0.117	0.073	0.046	0.133	0.120	0.074	0.044
HARQ3	$\hat{\alpha}$	0.560	0.794	0.922	0.960	0.606	0.824	0.932	0.962	0.616	0.820	0.912	0.954
	$\hat{L}$	0.192	0.170	0.105	0.065	0.133	0.116	0.072	0.046	0.134	0.120	0.074	0.044
	DM	1.165	-1.191	0.250	-1.818†	-0.155	-1.234	-1.687†	-0.239	1.042	-0.525	0.125	-0.438

Table 5: Relative performance of alternative out-of-sample forecasts of S&P500 and WTI Crude Oil futures realized volatility quantiles. For each model, quantile (α) and forecasts horizon (h), we report the unconditional coverage ($\hat{\alpha}$), the value of the tick-loss function ($\hat{L}$) and the Diebold-Mariano test statistic for equal predictive accuracy with the linear quantile regression model LQR2 serving as the benchmark. We use $*$ to denote significantly less accurate forecasts and †to denote significantly more accurate forecasts with respect to the benchmark at the 5% significance level.

References

Barndorff-Nielsen, O. E. & Shephard, N. (2004). Power and bipower variation with stochastic volatility and jumps, *Journal of Financial Econometrics* **2**(1): 1–48.

Corradi, V., Distaso, W. & Swanson, N. R. (2011). Predictive inference for integrated volatility, *Journal of the American Statistical Association* **106**: 1496–1512.