Econometrics Journal (2018), volume **01**, pp. S1–S31.
Article No. ectj??????

Quantile Coherency: A General Measure for Dependence between Cyclical Economic Variables: Online supplement

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Received: October 2018

S1. FURTHER QUANTITIES RELATED TO THE QUANTILE CROSS-SPECTRAL DENSITY KERNEL

In the situation described in this paper, there exists a right continuous orthogonal increment process $\{Z_j^\tau(\omega) : -\pi \leq \omega \leq \pi\}$, for every $j \in \{1, \dots, d\}$ and $\tau \in [0, 1]$, such that the Cramér representation

$$I\{X_{t,j} \leq q_j(\tau)\} = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}t\omega} \mathrm{d}Z_j^\tau(\omega)$$

holds [cf., e. g., Theorem 1.2.15 in Taniguchi and Kakizawa (2000)]. Note the fact that $(X_{t,j})_{t\in\mathbb{Z}}$ is strictly stationary and therefore $(I\{X_{t,j} \leq q_j(\tau)\})_{t\in\mathbb{Z}}$ is second-order stationary, as the boundedness of the indicator functions implies existence of their second moments.

The quantile cross-spectral density kernels are closely related to these orthogonal increment processes [cf. (Brillinger, 1975, p. 101) and (Brockwell and Davis, 1987, p. 436)]. More specifically, for $-\pi \leq \omega_1 \leq \omega_2 \leq \pi$, the following relation holds:

$$\int_{\omega_1}^{\omega_2} \mathfrak{f}^{j_1,j_2}(\omega; \tau_1, \tau_2)\mathrm{d}\omega = \mathrm{Cov}\left(Z_{j_1}^{\tau_1}(\omega_2) - Z_{j_1}^{\tau_1}(\omega_1), Z_{j_2}^{\tau_2}(\omega_2) - Z_{j_2}^{\tau_2}(\omega_1)\right),$$

or shortly: $\mathfrak{f}^{j_1,j_2}(\omega; \tau_1, \tau_2) = \mathrm{Cov}(\mathrm{d}Z_{j_1}^{\tau_1}(\omega), \mathrm{d}Z_{j_2}^{\tau_2}(\omega))$. It is important to observe that $\mathfrak{f}^{j_1,j_2}(\omega; \tau_1, \tau_2)$ is complex-valued. One way to represent $\mathfrak{f}^{j_1,j_2}(\omega; \tau_1, \tau_2)$ is to decompose it into its real and imaginary part. The real part is known as the cospectrum (of the processes $(I\{X_{t,j_1} \leq q_{j_1}(\tau_1)\})_{t\in\mathbb{Z}}$ and $(I\{X_{t,j_2} \leq q_{j_2}(\tau_2)\})_{t\in\mathbb{Z}}$). The negative of the imaginary part is commonly referred to as the quadrature spectrum. We will refer to these quantities as the quantile cospectrum and quantile quadrature spectrum of $(X_{t,j_1})_{t\in\mathbb{Z}}$ and $(X_{t,j_2})_{t\in\mathbb{Z}}$. Occasionally, to emphasise that these spectra are functions of (τ_1, τ_2), we will refer to them as the quantile cospectrum kernel and quantile quadrature spectrum kernel, respectively. The quantile quadrature spectrum vanishes if $j_1 = j_2$ and $\tau_1 = \tau_2$. More generally, as described in Kley et al. (2016), for any fixed j_1, j_2, the quadrature spectrum will vanish, for all τ_1, τ_2, if and only if (X_{t-k,j_1}, X_{t,j_2}) and (X_{t+k,j_1}, X_{t,j_2}) possess the same copula, for all k.

Table S.1. Spectral quantities related to $\mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2)$.

Name	Symbol
quantile cospectrum of $(X_{t,j_1})_{t\in\mathbb{Z}}$ and $(X_{t,j_2})_{t\in\mathbb{Z}}$	$\Re\mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2)$
quantile quadrature spectrum of $(X_{t,j_1})_{t\in\mathbb{Z}}$ and $(X_{t,j_2})_{t\in\mathbb{Z}}$	-$\Im\mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2)$
quantile amplitude spectrum of $(X_{t,j_1})_{t\in\mathbb{Z}}$ and $(X_{t,j_2})_{t\in\mathbb{Z}}$	$\lvert\mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2)\rvert$
quantile phase spectrum of $(X_{t,j_1})_{t\in\mathbb{Z}}$ and $(X_{t,j_2})_{t\in\mathbb{Z}}$	$\arg(\mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2))$
quantile coherency of $(X_{t,j_1})_{t\in\mathbb{Z}}$ and $(X_{t,j_2})_{t\in\mathbb{Z}}$	$\mathfrak{R}^{j_1,j_2}(\omega;\tau_1,\tau_2)$
quantile coherence of $(X_{t,j_1})_{t\in\mathbb{Z}}$ and $(X_{t,j_2})_{t\in\mathbb{Z}}$	$\lvert\mathfrak{R}^{j_1,j_2}(\omega;\tau_1,\tau_2)\rvert^2$

Note: The quantile cross-spectral density kernel $\mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2)$ of $(X_{t,j_1})_{t\in\mathbb{Z}}$ and $(X_{t,j_2})_{t\in\mathbb{Z}}$ is defined in (**??**).

An alternative way to look at $\mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2)$ is by representing it in polar coordinates. The radius $|\mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2)|$ is then referred to as the amplitude spectrum (of the two processes $(I\{X_{t,j_1} \le q_{j_1}(\tau_1)\})_{t\in\mathbb{Z}}$ and $(I\{X_{t,j_2} \le q_{j_2}(\tau_2)\})_{t\in\mathbb{Z}}$), while the angle $\arg(\mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2))$ is the so called phase spectrum, respectively. We refer to these quantities as the quantile amplitude spectrum and the quantile phase spectrum of $(X_{t,j_1})_{t\in\mathbb{Z}}$ and $(X_{t,j_2})_{t\in\mathbb{Z}}$. We note that the quantile spectral distribution function $\int_0^\omega \mathfrak{f}^{j_1,j_2}(\lambda;\tau_1,\tau_2))\mathrm{d}\lambda$ is clearly another way to represent the quantile-based dependence in the frequency domain. Its properties and estimation procedures are currently investigated in a separate research project and therefore not further discussed here.

Note that quantile coherency $\mathfrak{R}^{j_1,j_2}(\omega;\tau_1,\tau_2)$which we defined in Section **??** as a measure for the dynamic dependence of the two processes $(X_{t,j_1})_{t\in\mathbb{Z}}$ and $(X_{t,j_2})_{t\in\mathbb{Z}}$ is the correlation between $\mathrm{d}Z_{j_1}^{\tau_1}(\omega)$ and $\mathrm{d}Z_{j_2}^{\tau_2}(\omega)$. Its modulus squared $|\mathfrak{R}^{j_1,j_2}(\omega;\tau_1,\tau_2)|^2$ is referred to as the quantile coherence kernel of $(X_{t,j_1})_{t\in\mathbb{Z}}$ and $(X_{t,j_2})_{t\in\mathbb{Z}}$. A value of $|\mathfrak{R}^{j_1,j_2}(\omega;\tau_1,\tau_2)|$ close to 1 indicates a strong (linear) relationship between $\mathrm{d}Z_{j_1}^{\tau_1}(\omega)$ and $\mathrm{d}Z_{j_2}^{\tau_2}(\omega)$.

For the readers convenience, a list of the quantities and symbols introduced in this section is provided in Table S.1.

Estimators for the quantile cospectrum, quantile quadrature spectrum, quantile amplitude spectrum, quantile phase spectrum, and quantile coherence are then naturally given by $\Re\hat{G}_{n,R}^{j_1,j_2}(\omega;\tau_1,\tau_2)$, $-\Im\hat{G}_{n,R}^{j_1,j_2}(\omega;\tau_1,\tau_2)$, $|\hat{G}_{n,R}^{j_1,j_2}(\omega;\tau_1,\tau_2)|$, $\arg(\hat{G}_{n,R}^{j_1,j_2}(\omega;\tau_1,\tau_2))$, and $|\hat{\mathfrak{R}}_{n,R}^{j_1,j_2}(\omega;\tau_1,\tau_2)|^2$, respectively.

S2. AN EXAMPLE OF A PROCESS GENERATING QUANTILE DEPENDENCE ACROSS FREQUENCIES: QVAR(P)

For a better understanding of the dependence structures that we study in this paper, it is illustrative to introduce a process capable of generating them. We focus on generating dependence at different points of the joint distribution, which will vary across frequencies, but stays hidden from classical measures. In other words, we illustrate the intuition of spuriously independent variables, a situation when two variables seem to be independent when traditional cross-spectral analysis is used, while they are indeed clearly dependent at different parts of their joint distribution.

We base our example on a multivariate generalisation of the popular quantile autoregression process (QAR) introduced by Koenker and Xiao (2006). Inspired by vector autoregression processes (VAR), we link multiple QAR processes through their lag structure and refer to the resulting process as a quantile vector autoregression process (QVAR). This provides a natural way of generating rich dependence structure between two random variables in points of their joint distribution and over different frequencies. The autocovariance function of a stationary QVAR(p) process is that of a fixed parameter VAR(p) process. This follows from the argument by Knight (2006), who concludes that the exclusive use of autocorrelations may thus "fail to identify structure in the data that is potentially very informative". We will show how quantile spectral analysis reveals what otherwise may remain invisible.

Let $\boldsymbol{X}_t = (X_{t,1}, \dots, X_{t,d})'$, $t \in \mathbb{Z}$, be a sequence of random vectors that fulfills

$$\boldsymbol{X}_t = \sum_{j=1}^{p} \boldsymbol{\Theta}^{(j)}(\boldsymbol{U}_t)\boldsymbol{X}_{t-j} + \boldsymbol{\theta}^{(0)}(\boldsymbol{U}_t), \tag{S.1}$$

where $\boldsymbol{\Theta}^{(1)}, \dots, \boldsymbol{\Theta}^{(p)}$ are $d \times d$ matrices of functions, $\boldsymbol{\theta}^{(0)}$ is a $d \times 1$ column vector of functions, and $\boldsymbol{U}_t = (U_{t,1}, \dots, U_{t,d})'$, $t \in \mathbb{Z}$, is a sequence of independent vectors, with components $U_{t,k}$ that are $\mathcal{U}[0,1]$-distributed. We will assume that the elements of the ℓth row $\boldsymbol{\theta}^{(j)}_\ell(u_\ell) = \big(\theta^{(j)}_{\ell,1}(u_\ell), \dots, \theta^{(j)}_{\ell,d}(u_\ell)\big)$ of $\boldsymbol{\Theta}^{(j)}(u_1, \dots, u_d) = \big(\boldsymbol{\theta}^{(j)}_1(u_1)', \dots, \boldsymbol{\theta}^{(j)}_d(u_d)'\big)'$ and that the ℓth element $\theta^{(0)}_\ell(u_\ell)$ of $\boldsymbol{\theta}^{(0)} = \big(\theta^{(0)}_1(u_1), \dots, \theta^{(0)}_d(u_d)\big)'$ only depend on the ℓth variable, respectively. Under this assumption we can rewrite (S.1) as

$$X_{t,i} = \sum_{j=1}^{p} \boldsymbol{\theta}^{(j)}_i(U_{t,i})\boldsymbol{X}_{t-j} + \theta^{(0)}_i(U_{t,i}). \tag{S.2}$$

If the right hand side of (S.2) is monotonically increasing, then the conditional quantile function of $X_{t,i}$ given $(\boldsymbol{X}_{t-1}, \dots, \boldsymbol{X}_{t-p})$ can be represented as

$$Q_{X_{t,i}}(\tau | \boldsymbol{X}_{t-1}, \dots, \boldsymbol{X}_{t-p}) = \sum_{j=1}^{p} \boldsymbol{\theta}^{(j)}_i(\tau)\boldsymbol{X}_{t-j} + \theta^{(0)}_i(\tau).$$

Note that in this design the ℓth component of $\boldsymbol{U}_t$ determines the coefficients for the autoregression equation of the ℓth component of $\boldsymbol{X}_t$. We refer to the process as a quantile vector autoregression process of order p, hence QVAR(p). The class of processes (S.1) without assumptions regarding the parameters $\boldsymbol{\Theta}^{(j)}$ is naturally richer. Yet, the interpretation of the parameters in terms of the conditional quantile functions is possibly lost.

In the bivariate case ($d = 2$) of order $p = 1$, i.e. QVAR(1), (S.1) takes the following form:

$$\begin{pmatrix} X_{t,1} \\ X_{t,2} \end{pmatrix} = \begin{pmatrix} \theta^{(1)}_{11}(U_{t,1}) & \theta^{(1)}_{12}(U_{t,1}) \\ \theta^{(1)}_{21}(U_{t,2}) & \theta^{(1)}_{22}(U_{t,2}) \end{pmatrix} \begin{pmatrix} X_{t-1,1} \\ X_{t-1,2} \end{pmatrix} + \begin{pmatrix} \theta^{(0)}_1(U_{t,1}) \\ \theta^{(0)}_2(U_{t,2}) \end{pmatrix}.$$

For the examples we assume that the components $U_{t,1}$ and $U_{t,2}$ are independent and set the components of $\boldsymbol{\theta}^{(0)}$ to $\theta^{(0)}_1(u) = \theta^{(0)}_2(u) = \Phi^{-1}(u)$, $u \in [0,1]$, where $\Phi^{-1}(u)$ denotes the u-quantile of the standard normal distribution. Further, we set the diagonal elements of of $\boldsymbol{\Theta}^{(1)}$ to zero (i. e., $\theta^{(1)}_{11}(u) = \theta^{(1)}_{22}(u) = 0$, $u \in [0,1]$) and the off-diagonal elements to $\theta^{(1)}_{12}(u) = \theta^{(1)}_{21}(u) = 1.2(u - 0.5)$, $u \in [0,1]$. We thus create cross-dependence by linking the two processes with each other through the other ones lagged contributions. Note that this particular choice of parameter functions leads to the existence of a unique, strictly

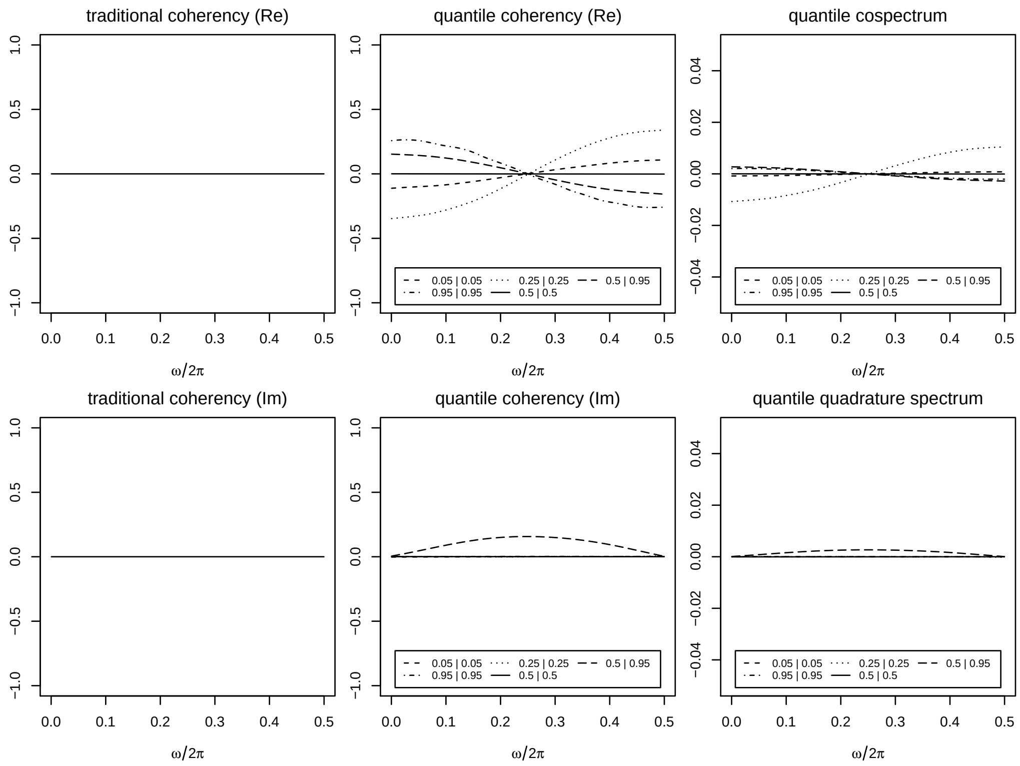

Figure S.1. Example of dependence structures generated by QVAR(1).

stationary solution; cf. Bougerol and Picard (1992). $(X_{t,1})_{t\in\mathbb{Z}}$ and $(X_{t,2})_{t\in\mathbb{Z}}$ are uncorrelated. Note that Hafner and Linton (2006) discuss that univariate quantile autoregression nests the popular autoregressive conditional heteroskedasticity (ARCH) models in terms of second order properties. Analogously, our QVAR(1) can be seen to nest a multivariate versions of ARCH.

In Figure S.1 the dynamics of the described QVAR(1) process are depicted. In terms of traditional coherency there appears to be no dependence across all frequencies. In terms of quantile coherency, on the other hand, rich dynamics are revealed in the different parts of the joint distribution. While, in the centre of the distribution (at the 0.5|0.5 level) the dependence is zero across frequencies, we see that the dependence increases if at least one of the quantile levels (τ_1, τ_2) is chosen closer to 0 or 1. More precisely, we see that the quantile coherency of this QVAR process resembles the shape of an VAR(1) process with coefficient matrix $\mathbf{\Theta}^{(1)}(\tau_1, \tau_2)$. The two processes are, for example when $\tau_1 = 0.05$ and $\tau_2 = 0.95$, clearly positively connected at lower frequencies with exactly the opposite value of quantile coherency at high frequencies, where the processes are in opposition. This also resembles the dynamics of the simple motivating examples from the introductory section of this paper, and highlights the importance of the quantile cross-spectral analysis as the dependence structure stays hidden if only the traditional measures are used.

In a second and third example, we consider a similar structure of parameters at the

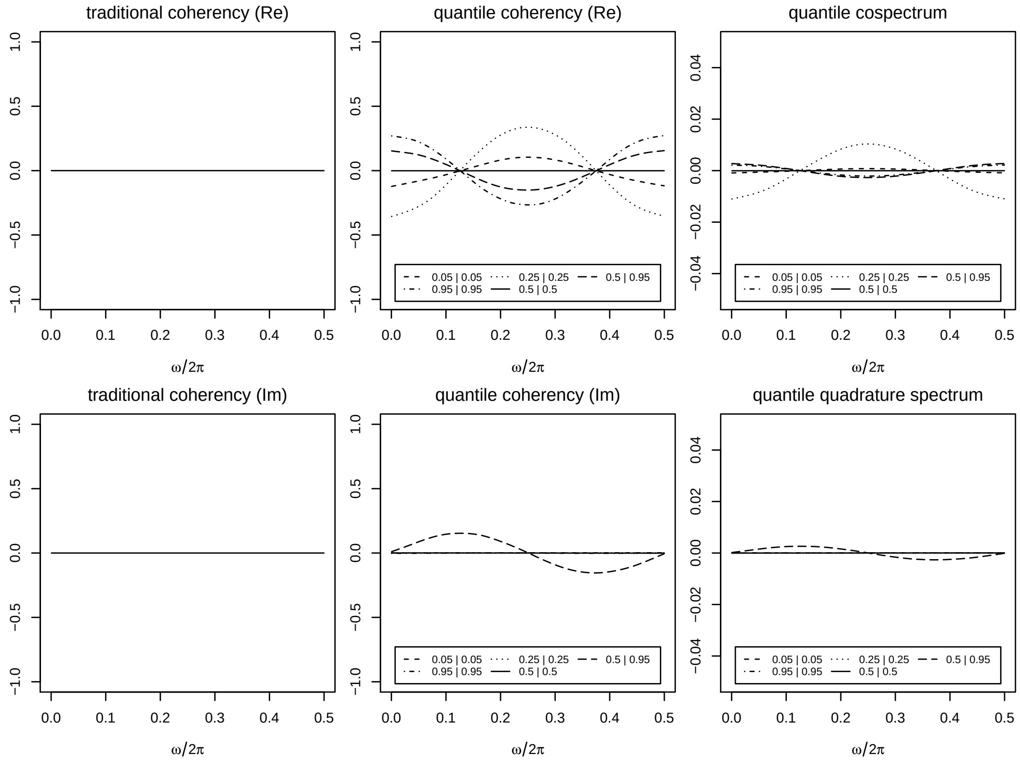

Figure S.2. Example of dependence structures generated by QVAR(2).

second and third lag. For the QVAR(2) process we let $\theta_{11}^{(j)}(u) = \theta_{22}^{(j)}(u) = 0$, for $j = 1, 2$, $\theta_{12}^{(1)}(u) = \theta_{21}^{(1)}(u) = 0$ and $\theta_{12}^{(2)}(u) = \theta_{21}^{(2)}(u) = 1.2(u - 0.5)$. In other words, here, the processes are connected through the second lag of the other one and, again, not directly through their own lagged contributions. In the QVAR(3) process, all coefficients are again set to zero, except for $\theta_{12}^{(3)}(u) = \theta_{21}^{(3)}(u) = 1.2(u - 0.5)$, such that the processes are connected only through the third lag of the other component and not through their own contributions.

In Figures S.2 and S.3 the dynamics of the described QVAR(2) and QVAR(3) processes are shown. Connecting the quantiles of the two processes through the second and third lag gives us richer dependence structures across frequencies. They, again, resemble the shape of the traditional coherencies of VAR(2) and VAR(3) processes. When traditional coherency is used for the QVAR(2) and QVAR(3) processes, the dependence structure stays completely hidden.

These examples of the general QVAR(p) specified in (S.1) served to show how rich dependence structures can be created across points of the joint distribution and different frequencies. It is obvious, how more complicated structures for the coefficient functions would lead to even richer dynamics than in the examples shown.

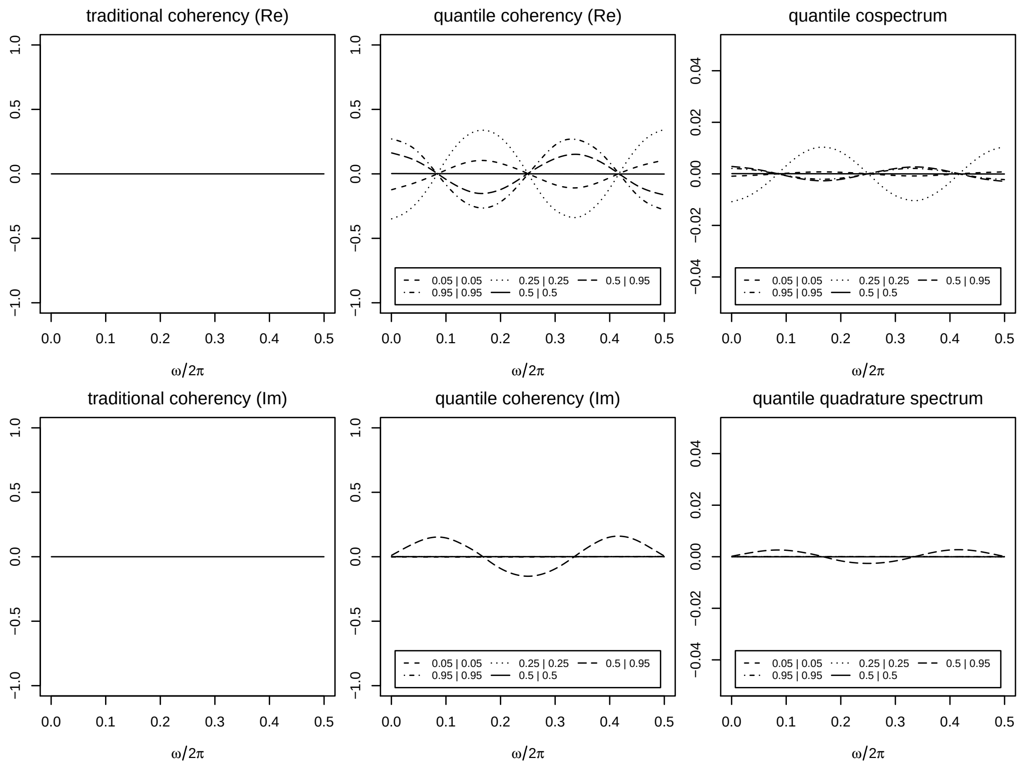

Figure S.3. Example of dependence structures generated by QVAR(3).

S3. RELATION BETWEEN QUANTILE AND TRADITIONAL SPECTRAL QUANTITIES IN THE CASE OF GAUSSIAN PROCESSES

When applying the proposed quantities, it is important to proceed with care when relating them to the traditional correlation and coherency measures. In this section we examine the case of a weakly stationary, multivariate process, where the proposed, quantile-based quantities and their traditional counterparts are directly related. The aim of the discussion is twofold. On one hand it provides assistance in how to interpret the quantile spectral quantities when the model is known to be Gaussian. On the other hand, and more importantly, it provides additional insight in how the traditional quantities break down when the serial dependency structure is not completely specified by the second moments.

We start by the discussion of the general case, where the process under consideration is assumed to be stationary, but needs not to be Gaussian. We will state conditions under which the traditional spectra (i. e., the matrix of spectral densities and cross-spectral densities) uniquely determines the quantile spectra (i. e., the matrix of quantile spectral densities and cross-spectral densities). In the end of this section we will discuss three examples of bivariate, stationary Gaussian processes and explain how the traditional coherency and the quantile coherency are related.

Denote by $\boldsymbol{c} := \{c_k^{j_1,j_2} : j_1, j_2 \in \{1,\ldots,d\}, k \in \mathbb{Z}\}$. $c_k^{j_1,j_2} := \mathrm{Cov}(X_{t+k,j_1}, X_{t,j_2})$, the family of auto- and cross-covariances. We will also refer to them as the second moment features of the process. We assume that $(|c_k^{j_1,j_2}|)_{k\in\mathbb{Z}}$ is summable, such that the traditional spectra $f^{j_1,j_2}(\omega) := (2\pi)^{-1}\sum_{k\in\mathbb{Z}} c_k^{j_1,j_2}\mathrm{e}^{-\mathrm{i}k\omega}$ exist. Because of the relation $c_k^{j_1,j_2} = \int_{-\pi}^{\pi} f^{j_1,j_2}(\omega)\mathrm{e}^{\mathrm{i}k\omega}\mathrm{d}\omega$ we will equivalently refer to $\boldsymbol{f}(\omega) := (f^{j_1,j_2}(\omega))_{j_1,j_2=1,\dots,d}$ as the second moment features of the process.

We now state conditions under which the traditional spectra uniquely determine the quantile spectra. Assume that the marginal distribution of $X_{t,j}$ $(j \in \{1,\dots,d\})$, which we denote by F_j, does not depend on t and is continuous. Further, the joint distribution of $\big(F_{j_1}(X_{t+k,j_1}), F_{j_2}(X_{t,j_2})\big)$, $j_1, j_2 \in \{1,\dots,d\}$, i. e. the copula of the pair (X_{t+k,j_1}, X_{t,j_2}), shall depend only on k, but not on t, and be uniquely specified by the second moment features of the process. More precisely, we assume the existence of functions $C_k^{j_1,j_2}$, such that

$$C_k^{j_1,j_2}\big(\tau_1,\tau_2;\boldsymbol{c}\big) = \mathbb{P}\big(F_{j_1}(X_{t+k,j_1}) \le \tau_1, F_{j_2}(X_{t,j_2}) \le \tau_2\big).$$

Obviously, $\mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2)$ is then, if it exists, uniquely determined by $\boldsymbol{c}$ [note (**??**) and the fact that $\gamma_k^{j_1,j_2}(\tau_1,\tau_2) = C_k^{j_1,j_2}\big(\tau_1,\tau_2;\boldsymbol{c}\big) - \tau_1\tau_2$].

In the case of stationary Gaussian processes the assumptions sufficient for the quantile spectra to be uniquely identified by the traditional spectra hold with

$$C_k^{j_1,j_2}\big(\tau_1,\tau_2;\boldsymbol{c}\big) := C^{\mathrm{Gauss}}(\tau_1,\tau_2; c_k^{j_1,j_2}(c_0^{j_1,j_1}c_0^{j_2,j_2})^{-1/2}),$$

where we have denoted the Gaussian copula by $C^{\mathrm{Gauss}}(\tau_1,\tau_2;\rho)$.

The converse can be stated under less restrictive conditions. If the marginal distributions are both known and both possess second moments, then the quantile spectra uniquely determine the traditional spectra.

Assume now the previously described situation in which the second moment features $\boldsymbol{f}$ uniquely determine the quantile spectra, which we denote by $\mathfrak{f}_{\boldsymbol{f}}^{j_1,j_2}(\omega;\tau_1,\tau_2)$ to stress the fact that it is determined by $\boldsymbol{f}$. Thus, the relation between the traditional spectra and the quantile spectra is 1-to-1. Denote the traditional coherency by $R^{j_1,j_2}(\omega) := f^{j_1,j_2}(\omega)/(f^{j_1,j_1}(\omega)f^{j_2,j_2}(\omega))^{1/2}$ and observe that it is also uniquely determined by the second moment features $\boldsymbol{f}$. Because the quantile coherency is determined by the quantile spectra which is related to the second moment features $\boldsymbol{f}$, as previously explained, we have established the relation of the traditional coherency and the quantile coherency. Obviously, this relation is not necessarily 1-to-1 anymore.

If the stationary process is from a parametric family of time series models the second moment features can be determined for each parameter. We now discuss three examples of Gaussian processes. Each example will have more complex serial dependence than the previous one. Without loss of generality we consider only bivariate examples. The first example is the one of non-degenerate Gaussian white noise. More precisely, we consider a Gaussian process $(X_{t,1}, X_{t,2})_{t\in\mathbb{Z}}$, where $\mathrm{Cov}(X_{t,i}, X_{s,j}) = 0$ and $\mathrm{Var}(X_{t,i}) > 0$, for all $t \neq s$ and $i, j \in \{1,2\}$.

Observe that, due to the independence of $(X_{t,1}, X_{t,2})$ and $(X_{s,1}, X_{s,2})$, $t \neq s$, we have $\gamma_k^{1,2}(\tau_1,\tau_2) = 0$ for all $k \neq 0$ and $\tau_1,\tau_2 \in [0,1]$. It is easy to see that

$$\mathfrak{R}^{1,2}(\omega;\tau_1,\tau_2) = \frac{C^{\mathrm{Gauss}}(\tau_1,\tau_2;R^{1,2}(\omega)) - \tau_1\tau_2}{\sqrt{\tau_1(1-\tau_1)}\sqrt{\tau_2(1-\tau_2))}} \tag{S.3}$$

where $R^{1,2}(\omega)$ denotes the traditional coherency, which in this case (a bivariate i. i. d. sequence) equals $c_0^{1,2}(c_0^{1,1}c_0^{2,2})^{-1/2}$ (for all ω).

By employing (S.3), we can thus determine the quantile coherency for any given traditional coherency and fixed combination of $\tau_1,\tau_2 \in (0,1)$. In the top-centre part of

Figure S.4 this conversion is visualised for four pairs of quantile levels and any possible traditional coherency. It is important to observe the limited range of the quantile coherency. For example, there never is strong positive dependence between the τ_1-quantile in the first component and the τ_2-quantile in the second component when both τ_1 and τ_2 are close to 0. Similarly, there never is strong negative dependence when one of the quantile levels is chosen close to 0 while the other one is chosen close to 1. This observation is not special for the Gaussian case, but holds for any sequence of pairwise independent bivariate random variables. Bounds that correspond to the case of perfect positive or perfect negative dependence (at the level of quantiles), can be derived from the Fréchet/Hoeffding bounds for copulas: in the case of serial independence quantile coherency is bounded by

$$\frac{\max\{\tau_1 + \tau_2 - 1, 0\} - \tau_1\tau_2}{\sqrt{\tau_1(1-\tau_1)}\sqrt{\tau_2(1-\tau_2))}} \leq \Re^{1,2}(\omega; \tau_1, \tau_2) \leq \frac{\min\{\tau_1, \tau_2\} - \tau_1\tau_2}{\sqrt{\tau_1(1-\tau_1)}\sqrt{\tau_2(1-\tau_2))}}.$$

Note that these bounds hold for any joint distribution of $(X_{t,i}, X_{t,j})$. In particular, the bound holds independent of the correlation.

In the top-left part of Figure S.4 traditional coherencies are shown for this example. Because no serial dependence is present, all coherencies are flat lines. Their level is equal to the correlation between the two components. In the top-right part of Figure S.4 the quantile coherency for the example is shown when the correlation is 0.6 (the corresponding coherency is marked with a bold line in the top-left figure). Note that for fixed τ_1 and τ_2 the value of the quantile coherency corresponds to the value in the top-centre figure where the vertical grey line and the corresponding graph intersect. The quantile coherency in the right part does not depend on the frequency, because in this example there is no serial dependence.

In the top-centre part of Figure S.4 it is important to observe that for traditional coherency 0 (i. e., when the components are independent, due to $(X_{t,1}, X_{t,2})$ being uncorrelated jointly Gaussian) quantile coherency is zero at all quantile levels.

In the next two examples we stay in the Gaussian framework, but introduce serial dependence. Consider a bivariate, stable VAR(1) process $\boldsymbol{X}_t = (X_{t,1}, X_{t,2})'$, $t \in \mathbb{Z}$, fulfilling the difference equation

$$\boldsymbol{X}_t = \boldsymbol{A}\boldsymbol{X}_{t-1} + \boldsymbol{\varepsilon}_t, \tag{S.4}$$

with parameter $\boldsymbol{A} \in \mathbb{R}^{2\times 2}$ and i. i. d., centred, bivariate, jointly normally distributed innovations $\boldsymbol{\varepsilon}_t$ with unit variance $E(\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t') = \boldsymbol{I}_2$.

In our second example serial dependence is introduced, by relating each component to the lagged other component in the regression equation. In other words, we consider model (S.4) where the matrix $\boldsymbol{A}$ has diagonal elements equal to 0 and some value a on the off-diagonal. Assuming $|a| < 1$ yields a stable process. As described earlier, the traditional spectral density matrix, which in this example is of the form

$$\boldsymbol{f}(\omega) := (2\pi)^{-1}\Big(\boldsymbol{I}_2 - \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \mathrm{e}^{-i\omega}\Big)^{-1}\Big(\boldsymbol{I}_2 - \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} \mathrm{e}^{i\omega}\Big)^{-1}, \ |a| < 1,$$

uniquely determines the traditional coherency and, because of the Gaussian innovations, also the quantile coherency.

In the middle-left plot of Figure S.4 the traditional coherencies for this model are shown when a takes different values. If we now fix a frequency $[\neq \pi/4]$, then the value of the traditional coherency for this frequency uniquely determines the value of a. In

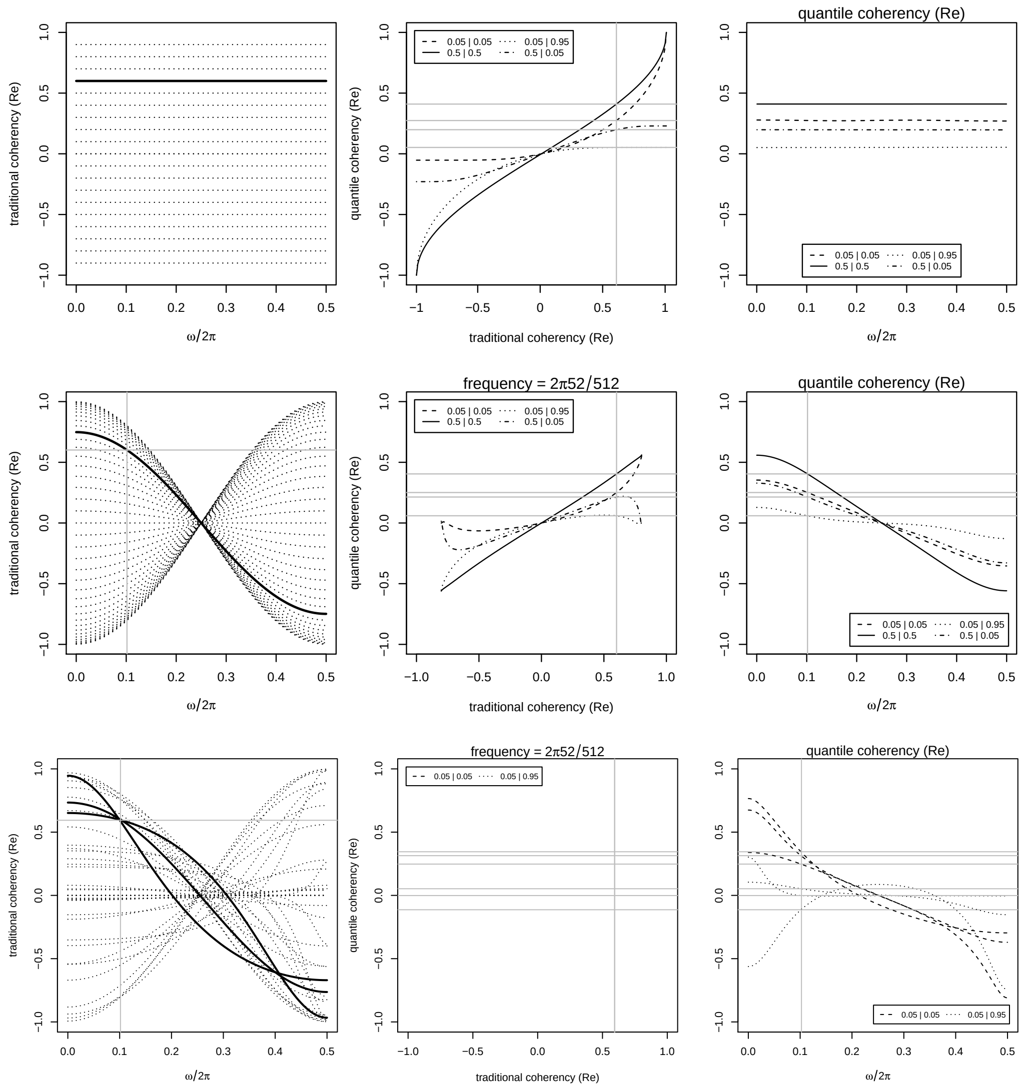

Figure S.4. Quantile and traditional coherency for selected Gaussian processes.

Figure S.4 we have marked the frequency of $\omega = 2\pi 52/512$ and coherency value of 0.6 by grey lines and printed the corresponding coherency (as a function of ω) in bold. Note that of the many pictured coherencies [one for each $a \in (-1, 1)$] only one has the value of 0.6 at this frequency. In the centre plot of the middle row we show the relation between the traditional coherency and quantile coherency for the considered model. For four combinations of quantile levels and all values of $a \in (-1, 1)$ the corresponding traditional coherencies and quantile coherencies are shown. It is important to observe that the relation is shown only for one frequency [$\omega = 2\pi 52/512$]. We observe that the range of values for the quantile coherency is limited and that the range depends on the combination of quantile levels and on the frequency. While this is quite similar to the first example where quantile coherency had to be bounded due to the Fréchet/Hoeffding bounds, we here also observe (for this particular model and frequency) that the range of values for the traditional coherency is limited. This fact is also apparent in the middle-left plot. To relate the traditional and quantile coherency at this particular frequency, one can, using the centre-middle plot, proceed as in the first example. For a given frequency choose a valid traditional coherency (x-axis of the middle-centre plot) and combination of quantile levels (one of the lines in the plot) and then determine the value for the quantile coherency (depicted in the right plot). Note that (in this example), for a given frequency and combination of quantile levels the relation is still a function of the traditional coherency, but fails to be injective.

In our final example we consider the Gaussian VAR(1) model (S.4) where we now allow for an additional degree of freedom, by letting the matrix $\boldsymbol{A}$ be of the form where the diagonal elements both are equal to b and keep the value a on the off-diagonal as before. Thus, compared to the previous example, where $b = 0$ was required, each component now may also depend on its own lagged value. It is easy to see that $|a + b| < 1$ yields a stable process. In this case the tradtional spectral density matrix is of the form

$$\boldsymbol{f}(\omega) := (2\pi)^{-1}\Big(\boldsymbol{I}_2 - \begin{pmatrix} b & a \\ a & b \end{pmatrix} \mathrm{e}^{-i\omega}\Big)^{-1}\Big(\boldsymbol{I}_2 - \begin{pmatrix} b & a \\ a & b \end{pmatrix} \mathrm{e}^{i\omega}\Big)^{-1}, \ |a + b| < 1.$$

In the bottom-left part of Figure S.4 a collection of traditional coherencies (as functions of ω) is shown. Due to the extra degree of freedom in the model the variety of shapes increased dramatically. In particular, for a given frequency, the value of the traditional coherency does not uniquely specify the model parameter any more. We have marked three coherencies (as functions of ω) that have value 0.6 at $\omega = 2\pi 52/512$ in bold to stress this fact. The corresponding processes have (for a fixed combination of quantile levels) different values of quantile coherency at this frequency. This fact can be seen from the bottom-centre part of Figure S.4, where the relation between traditional and quantile coherency is depicted for the frequency fixed and two combinations of quantile levels are shown in black and grey. Note the important fact that the relation (for fixed frequency) is not a function of the traditional coherency any more. The bottom-right part of the figure shows the quantile coherency curves (as a function of ω) for the three model parameters (shown in bold in the bottom-left part of the figure) and the two combination of quantile levels. It is clearly visible that even though, for the particular fixed frequency, the traditional coherency coincide, the value and shape of the quantile coherency can be very different depending on the underlying process. This third example illustrated how a frequency-by-frequency comparison of the traditional coherency with its quantile-based counterpart may fail, even when the process is quite simple.

We have seen, from the theoretical discussion in the beginning of this section, that for Gaussian processes, when the marginal distributions are fixed, a relation between the traditional spectra and the quantile spectra exists. This relation is a 1-to-1 relation between the quantities as functions of frequency (and quantile levels). The three examples have illustrated that a comparison on a frequency-by-frequency basis may be possible in special cases but does not hold in general.

In conclusion we therefore advise to see the quantile cross-spectral density as a measure for dependence on its own, as the quantile-based quantities focus on more general types of dependence. We further point out that quantile coherency may be used in examples where the conditions that make a relation possible are fulfilled, but also, for example, to analyse the dependence in the quantile vector autoregressive (QVAR) processes, described in Section S2. The QVAR processes possess more complicated dynamics, which cannot be described only by the second order moment features.

S4. ASYMPTOTIC PROPERTIES OF THE PROPOSED ESTIMATORS FOR QUANTILE CROSS-SPECTRAL DENSITIES

We are now going to state a result on the asymptotic properties of the CCR-periodogram $\boldsymbol{I}_{n,R}(\omega;\tau_1,\tau_2)$ defined in (**??**) and (**??**).

PROPOSITION S4.1. *Assume that $(\boldsymbol{X}_t)_{t\in\mathbb{Z}}$ is strictly stationary and satisfies Assumption* **??**. *Further assume that the marginal distribution functions F_j, $j = 1,\ldots,d$ are continuous. Then, for every fixed $\omega \neq 0 \mod 2\pi$,*

$$\Big(\boldsymbol{I}_{n,R}(\omega;\tau_1,\tau_2)\Big)_{(\tau_1,\tau_2)\in[0,1]^2} \Rightarrow \Big(\mathbb{I}(\omega;\tau_1,\tau_2)\Big)_{(\tau_1,\tau_2)\in[0,1]^2} \quad \text{in } \ell^\infty_{\mathbb{C}^{d\times d}}([0,1]^2). \tag{S.5}$$

The $\mathbb{C}^{d\times d}$-valued limiting processes $\mathbb{I}$, indexed by $(\tau_1,\tau_2)\in[0,1]^2$, is of the form

$$\mathbb{I}(\omega;\tau_1,\tau_2) = \frac{1}{2\pi}\mathbb{D}(\omega;\tau_1)\overline{\mathbb{D}(\omega;\tau_2)'},$$

where $\mathbb{D}(\omega;\tau) = (\mathbb{D}^j(\omega;\tau))_{j=1,\ldots,d}$, $\tau\in[0,1]$, $\omega\in\mathbb{R}$ is a centred, $\mathbb{C}^d$-valued Gaussian processes with covariance structure of the following form

$$\operatorname{Cov}(\mathbb{D}^{j_1}(\omega;\tau_1),\mathbb{D}^{j_2}(\omega;\tau_2)) = 2\pi\mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2).$$

Moreover, $\mathbb{D}(\omega;\tau) = \overline{\mathbb{D}(-\omega;\tau)} = \mathbb{D}(\omega+2\pi;\tau)$, and the family $\{\mathbb{D}(\omega;\cdot) \ : \ \omega\in[0,\pi]\}$ is a collection of independent processes. In particular, the weak convergence (S.5) holds jointly for any finite fixed collection of frequencies ω.

For $\omega = 0 \mod 2\pi$ the asymptotic behaviour of the CCR-periodogram is as follows: we have $d^j_{n,R}(0;\tau) = n\tau + o_p(n^{1/2})$, where the exact form of the remainder term depends on the number of ties in $X_{j,0},\ldots,X_{j,n-1}$. Therefore, under the assumptions of Proposition S4.1, we have $\boldsymbol{I}_{n,R}(0;\tau_1,\tau_2) = n(2\pi)^{-1}\tau_1\tau_2 1_d 1_d' + o_p(1)$, where $1_d := (1,\ldots,1)'\in\mathbb{R}^d$.

We now state a result that quantifies the uncertainty in estimating $\mathfrak{f}(\omega;\tau_1,\tau_2)$ by $\boldsymbol{G}_{n,R}(\omega;\tau_1,\tau_2)$ asymptotically.

THEOREM S4.1. *Let Assumptions* **??** *and* **??** *hold. Assume that the marginal distribution functions F_j, $j=1,\ldots,d$ are continuous and that constants $\kappa>0$ and $k\in\mathbb{N}$ exist, such that $b_n = o(n^{-1/(2k+1)})$ and $b_n n^{1-\kappa}\to\infty$. Then, for any fixed $\omega\in\mathbb{R}$, the process*

$$\mathbb{G}_n(\omega;\cdot,\cdot) := \sqrt{nb_n}\Big(\hat{\boldsymbol{G}}_{n,R}(\omega;\tau_1,\tau_2) - \mathfrak{f}(\omega;\tau_1,\tau_2) - \boldsymbol{B}_n^{(k)}(\omega;\tau_1,\tau_2)\Big)_{\tau_1,\tau_2\in[0,1]}$$

satisfies

$$\mathbb{G}_n(\omega;\cdot,\cdot) \Rightarrow \mathbb{H}(\omega;\cdot,\cdot) \ \text{in } \ell^\infty_{\mathbb{C}^{d\times d}}([0,1]^2), \tag{S.6}$$

where the elements of the bias matrix $\boldsymbol{B}_n^{(k)}$ are given by

$$\Big\{\boldsymbol{B}_n^{(k)}(\omega;\tau_1,\tau_2)\Big\}_{j_1,j_2} := \sum_{\ell=2}^{k}\frac{b_n^\ell}{\ell!}\int_{-\pi}^{\pi} v^\ell W(v)dv\frac{\mathrm{d}^\ell}{\mathrm{d}\omega^\ell}\mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2) \tag{S.7}$$

and $\mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2)$ is defined in (**??**). *The process $\mathbb{H}(\omega;\cdot,\cdot) := (\mathbb{H}^{j_1,j_2}(\omega;\cdot,\cdot))_{j_1,j_2=1,\ldots,d}$*

in (S.6) *is a centred,* $\mathbb{C}^{d\times d}$*-valued Gaussian process characterised by*

$$\begin{aligned}\mathrm{Cov}\left(\mathbb{H}^{j_1,j_2}(\omega;u_1,v_1),\mathbb{H}^{k_1,k_2}(\lambda;u_2,v_2)\right)\\ = 2\pi\Big(\int_{-\pi}^{\pi}W^2(\alpha)\mathrm{d}\alpha\Big)\Big(\mathfrak{f}^{j_1,k_1}(\omega;u_1,u_2)\mathfrak{f}^{j_2,k_2}(-\omega;v_1,v_2)\eta(\omega-\lambda)\\ +\mathfrak{f}^{j_1,k_2}(\omega;u_1,v_2)\mathfrak{f}^{j_2,k_1}(-\omega;v_1,u_2)\eta(\omega+\lambda)\Big),\end{aligned} \quad (S.8)$$

where $\eta(x) := I\{x = 0(\mod 2\pi)\}$ *[cf. (Brillinger, 1975, p. 148)] is the* 2π*-periodic extension of Kronecker's delta function. The family* $\{\mathbb{H}(\omega;\cdot,\cdot),\ \omega\in[0,\pi]\}$ *is a collection of independent processes and* $\mathbb{H}(\omega;\tau_1,\tau_2)=\overline{\mathbb{H}(-\omega;\tau_1,\tau_2)}=\mathbb{H}(\omega+2\pi;\tau_1,\tau_2)$.

A few remarks on the result are in order. In sharp contrast to classical spectral analysis, where higher-order moments are required to obtain smoothness of the spectral density [cf. Brillinger (1975), p. 27], Assumption **??** guarantees that the quantile cross-spectral density is an analytical function of ω. Hence, the kth derivative of $\omega\mapsto\mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2)$ in (S.7) exists without further assumptions.

The case $\omega = 0 \mod 2\pi$ does not require separate treatment as in Proposition S4.1, because $I^{j_1,j_2}_{n,R}(0,\tau_1,\tau_2)$ is excluded in (**??**): the definition of $\hat{G}^{j_1,j_2}_{n,R}(\omega;\tau_1,\tau_2)$.

Assume that W is a kernel of order p; i. e., for some p, satisfies $\int_{-\pi}^{\pi}v^jW(v)\mathrm{d}v=0$, for all $j<p$, and $0<\int_{-\pi}^{\pi}v^pW(v)\mathrm{d}v<\infty$. E. g., the Epanechnikov kernel is a kernel of order $p=2$. Then, the bias is of order b_n^p. As the variance is of order $(nb_n)^{-1}$, the mean squared error is minimal, if $b_n\asymp n^{-1/(2p+1)}$. This optimal bandwidth fulfills the assumptions of Theorem S4.1. A detailed discussion of how Theorem S4.1 can be used to construct asymptotically valid confidence intervals is deferred to Section D.

The independence of the limit $\{\mathbb{H}(\omega;\cdot,\cdot),\ \omega\in[0,\pi]\}$ has two important implications. On one hand, the weak convergence (S.6) holds jointly for any finite fixed collection of frequencies ω. On the other hand, if one were to consider the smoothed CCR-periodogram as a function of the three arguments (ω,τ_1,τ_2), weak convergence cannot hold any more. This limitation of convergence is due to the fact that there exists no tight element in $\ell^\infty_{\mathbb{C}^{d\times d}}([0,\pi]\times[0,1]^2)$ that has the right finite-dimensional distributions, which would be required for process convergence in $\ell^\infty_{\mathbb{C}^{d\times d}}([0,\pi]\times[0,1]^2)$.

Fixing j_1,j_2 and τ_1,τ_2 the CCR-periodogram $\hat{G}^{j_1,j_2}_{n,R}(\omega;\tau_1,\tau_2)$ and traditional smoothed cross-periodogram determined from the unobservable, bivariate time series

$$\big(I\{F_{j_1}(X_{t,j_1})\le\tau_1\},I\{F_{j_1}(X_{t,j_2})\le\tau_2\}\big),\quad t=0,\dots,n-1, \quad (S.9)$$

are asymptotically equivalent. Theorem S4.1 thus reveals that in the context of the estimation of the quantile cross-spectral density the estimation of the marginal distribution has no impact on the limit distribution (cf. comment after Remark 3.5 in Kley et al. (2016)).

S5. ON THE CONSTRUCTION OF INTERVAL ESTIMATORS

In this section we collect details on how to construct pointwise confidence bands.

Sections **??** and S4 contained asymptotic results on the uncertainty of point estimation of the newly introduced quantile cross-spectral quantities. In this section we describe strategies to estimate the variances (of the real and imaginary parts) that appear in those limit results and describe how asymptotically valid pointwise confidence bands can be constructed.

In all three subsections the following comment is relevant. Assuming that we have determined the weights W_n form a kernel W that is of order d. We will choose a bandwidth $b_n = o(n^{-1/(2d+1)})$. This choice implies that compared to the variance the bias (that in some form appears in both limit results) is asymptotically negligible: $\sqrt{nb_n}\boldsymbol{B}_n^{(k)}(\omega;\tau_1,\tau_2) = o(1)$.

S5.1. Pointwise confidence bands for $\mathfrak{f}$

Utilising Theorem S4.1 we now construct pointwise asymptotic $(1-\alpha)$-level confidence bands for the real and imaginary parts of $\mathfrak{f}^{j_1,j_2}(\omega_{kn};\tau_1,\tau_2)$, $\omega_{kn} := 2\pi k/n$, as follows:

$$C_{\mathrm{r},n}^{(1)}(\omega_{kn};\tau_1,\tau_2) := \Re\tilde{G}_{n,R}^{j_1,j_2}(\omega_{kn};\tau_1,\tau_2) \pm \Re\sigma_{(1)}^{j_1,j_2}(\omega_{kn};\tau_1,\tau_2)\Phi^{-1}(1-\alpha/2),$$

for the real part, and

$$C_{\mathrm{i},n}^{(1)}(\omega_{kn};\tau_1,\tau_2) := \Im\tilde{G}_{n,R}^{j_1,j_2}(\omega_{kn};\tau_1,\tau_2) \pm \Im\sigma_{(1)}^{j_1,j_2}(\omega_{kn};\tau_1,\tau_2)\Phi^{-1}(1-\alpha/2),$$

for the imaginary part of the quantile cross-spectrum. Here,

$$\tilde{G}_{n,R}^{j_1,j_2}(\omega_{kn};\tau_1,\tau_2) := \hat{G}_{n,R}^{j_1,j_2}(\omega_{kn};\tau_1,\tau_2)/W_n^k, \quad W_n^k := \frac{2\pi}{n}\sum_{s=1}^{n-1} W_n(\omega_{kn}-\omega_{sn}),$$

and Φ denotes the cumulative distribution function of the standard normal distribution,[1]

$$\big(\Re\sigma^{j_1,j_2}(\omega_{kn};\tau_1,\tau_2)\big)^2 := 0 \vee \begin{cases} \mathrm{Cov}(\mathbb{H}_{1,2},\mathbb{H}_{1,2}) & \text{if } j_1 = j_2 \text{ and } \tau_1 = \tau_2, \\ \frac{1}{2}\big(\mathrm{Cov}(\mathbb{H}_{1,2},\mathbb{H}_{1,2}) + \Re\,\mathrm{Cov}(\mathbb{H}_{1,2},\mathbb{H}_{2,1})\big) & \text{otherwise,} \end{cases}$$

and

$$\big(\Im\sigma^{j_1,j_2}(\omega_{kn};\tau_1,\tau_2)\big)^2 := 0 \vee \begin{cases} 0 & \text{if } j_1 = j_2 \text{ and } \tau_1 = \tau_2, \\ \frac{1}{2}\big(\mathrm{Cov}(\mathbb{H}_{1,2},\mathbb{H}_{1,2}) - \Re\,\mathrm{Cov}(\mathbb{H}_{1,2},\mathbb{H}_{2,1})\big) & \text{otherwise,} \end{cases}$$

where $\mathrm{Cov}(\mathbb{H}_{a,b},\mathbb{H}_{c,d})$ denotes an estimator of $\mathrm{Cov}\big(\mathbb{H}^{j_a,j_b}(\omega_{kn};\tau_a,\tau_b),\mathbb{H}^{j_c,j_d}(\omega_{kn};\tau_c,\tau_d)\big)$. Here, motivated by Theorem 7.4.3 in Brillinger (1975), we use

$$\Big(\frac{2\pi}{n\cdot W_n^k}\Big)\times\Bigg[\sum_{s=1}^{n-1} W_n\big(2\pi(k-s)/n\big)W_n\big(2\pi(k-s)/n\big)\tilde{G}_{n,R}^{j_a,j_c}(\tau_a,\tau_c;2\pi s/n)\tilde{G}_{n,R}^{j_b,j_d}(\tau_b,\tau_d;-2\pi s/n)$$
$$+\sum_{s=1}^{n-1} W_n\big(2\pi(k-s)/n\big)W_n\big(2\pi(k+s)/n\big)\tilde{G}_{n,R}^{j_a,j_d}(\tau_a,\tau_d;2\pi s/n)\tilde{G}_{n,R}^{j_b,j_c}(\tau_b,\tau_c;-2\pi s/n)\Bigg] \tag{S.10}$$

The definition of $\sigma_{(1)}^{j_1,j_2}(\omega_{kn};\tau_1,\tau_2)$ is motivated by the fact that $\Im\hat{G}_{n,R}^{j_1,j_2}(\omega_{kn};\tau_1,\tau_2) = 0$, if $j_1 = j_2$ and $\tau_1 = \tau_2$. Furthermore, note that, for any complex-valued random variable Z, with complex conjugate $\bar{Z}$,

$$\mathrm{Var}(\Re Z) = \frac{1}{2}\big(\mathrm{Var}(Z) + \Re\,\mathrm{Cov}(Z,\bar{Z})\big); \quad \mathrm{Var}(\Im Z) = \frac{1}{2}\big(\mathrm{Var}(Z) - \Re\,\mathrm{Cov}(Z,\bar{Z})\big), \tag{S.11}$$

[1] Note that for $k = 0,\ldots,n-1$ we have $W_n^k := 2\pi/n\sum_{0=s\neq k} W_n(2\pi s/n)$. For $k\in\mathbb{Z}$ with $k<0$ or $k\geq n$ we can define it as the n periodic extension.

and we have $\overline{\mathbb{H}_{1,2}} = \mathbb{H}_{2,1}$.

S5.2. Pointwise confidence bands for $\mathfrak{R}$

We utilise Theorem ?? to construct pointwise asymptotic $(1-\alpha)$-level confidence bands for the real and imaginary parts of $\mathfrak{R}^{j_1,j_2}(\omega;\tau_1,\tau_2)$ as follows:

$$C^{(2)}_{\mathrm{r},n}(\omega_{kn};\tau_1,\tau_2) := \Re\hat{\mathfrak{R}}^{j_1,j_2}_{n,R}(\omega_{kn};\tau_1,\tau_2) \pm \Re\sigma^{j_1,j_2}_{(2)}(\omega_{kn};\tau_1,\tau_2)\Phi^{-1}(1-\alpha/2),$$

for the real part, and

$$C^{(2)}_{\mathrm{i},n}(\omega_{kn};\tau_1,\tau_2) := \Im\hat{\mathfrak{R}}^{j_1,j_2}_{n,R}(\omega_{kn};\tau_1,\tau_2) \pm \Im\sigma^{j_1,j_2}_{(2)}(\omega_{kn};\tau_1,\tau_2)\Phi^{-1}(1-\alpha/2),$$

for the imaginary part of the quantile coherency. Here, Φ stands for the cdf of the standard normal distribution,

$$\big(\Re\sigma^{j_1,j_2}_{(2)}(\omega_{kn};\tau_1,\tau_2)\big)^2 := 0 \vee \begin{cases} 0 & \text{if } j_1 = j_2 \\ & \text{and } \tau_1=\tau_2, \\ \frac{1}{2}\big(\operatorname{Cov}(\mathbb{L}_{1,2},\mathbb{L}_{1,2}) + \Re\operatorname{Cov}(\mathbb{L}_{1,2},\mathbb{L}_{2,1})\big) & \text{otherwise,} \end{cases}$$

and

$$\big(\Im\sigma^{j_1,j_2}_{(2)}(\omega_{kn};\tau_1,\tau_2)\big)^2 := 0 \vee \begin{cases} 0 & \text{if } j_1 = j_2 \\ & \text{and } \tau_1=\tau_2, \\ \frac{1}{2}\big(\operatorname{Cov}(\mathbb{L}_{1,2},\mathbb{L}_{1,2}) - \Re\operatorname{Cov}(\mathbb{L}_{1,2},\mathbb{L}_{2,1})\big) & \text{otherwise.} \end{cases}$$

The definition of $\sigma^{j_1,j_2}_{(2)}(\omega_{kn};\tau_1,\tau_2)$ is motivated by (S.11) and the fact that we have $\overline{\mathbb{L}_{1,2}} = \mathbb{L}_{2,1}$. Furthermore, note that $\hat{\mathfrak{R}}^{j_1,j_2}_{n,R}(\omega_{kn};\tau_1,\tau_2)=1$, if $j_1=j_2$ and $\tau_1=\tau_2$..
In the definition of $\sigma^{j_1,j_2}_{(2)}(\omega_{kn};\tau_1,\tau_2)$ we have used $\operatorname{Cov}(\mathbb{L}_{a,b},\mathbb{L}_{c,d})$ to denote an estimator for

$$\operatorname{Cov}\big(\mathbb{L}^{j_1,j_2}(\omega_{kn};\tau_1,\tau_2), \mathbb{L}^{j_3,j_4}(\omega_{kn};\tau_3,\tau_4)\big).$$

Recalling the definition of he limit process in Theorem ?? we derive the following expression:

$$\begin{aligned}
&\frac{1}{\sqrt{\mathfrak{f}_{1,1}\mathfrak{f}_{2,2}\mathfrak{f}_{3,3}\mathfrak{f}_{4,4}}}\operatorname{Cov}\Big(\mathbb{H}_{1,2} - \frac{1}{2}\frac{\mathfrak{f}_{1,2}}{\mathfrak{f}_{1,1}}\mathbb{H}_{1,1} - \frac{1}{2}\frac{\mathfrak{f}_{1,2}}{\mathfrak{f}_{2,2}}\mathbb{H}_{2,2}, \mathbb{H}_{3,4} - \frac{1}{2}\frac{\mathfrak{f}_{3,4}}{\mathfrak{f}_{3,3}}\mathbb{H}_{3,3} - \frac{1}{2}\frac{\mathfrak{f}_{3,4}}{\mathfrak{f}_{4,4}}\mathbb{H}_{4,4}\Big) \\
&= \frac{\operatorname{Cov}(\mathbb{H}_{1,2},\mathbb{H}_{3,4})}{\sqrt{\mathfrak{f}_{1,1}\mathfrak{f}_{2,2}\mathfrak{f}_{3,3}\mathfrak{f}_{4,4}}} - \frac{1}{2}\frac{\overline{\mathfrak{f}_{3,4}}\operatorname{Cov}(\mathbb{H}_{1,2},\mathbb{H}_{3,3})}{\sqrt{\mathfrak{f}_{1,1}\mathfrak{f}_{2,2}\mathfrak{f}^3_{3,3}\mathfrak{f}_{4,4}}} - \frac{1}{2}\frac{\overline{\mathfrak{f}_{3,4}}\operatorname{Cov}(\mathbb{H}_{1,2},\mathbb{H}_{4,4})}{\sqrt{\mathfrak{f}_{1,1}\mathfrak{f}_{2,2}\mathfrak{f}_{3,3}\mathfrak{f}^3_{4,4}}} \\
&\quad - \frac{1}{2}\frac{\mathfrak{f}_{1,2}\operatorname{Cov}(\mathbb{H}_{1,1},\mathbb{H}_{3,4})}{\sqrt{\mathfrak{f}^3_{1,1}\mathfrak{f}_{2,2}\mathfrak{f}_{3,3}\mathfrak{f}_{4,4}}} + \frac{1}{4}\frac{\mathfrak{f}_{1,2}\overline{\mathfrak{f}_{3,4}}\operatorname{Cov}(\mathbb{H}_{1,1},\mathbb{H}_{3,3})}{\sqrt{\mathfrak{f}^3_{1,1}\mathfrak{f}_{2,2}\mathfrak{f}^3_{3,3}\mathfrak{f}_{4,4}}} + \frac{1}{4}\frac{\mathfrak{f}_{1,2}\overline{\mathfrak{f}_{3,4}}\operatorname{Cov}(\mathbb{H}_{1,1},\mathbb{H}_{4,4})}{\sqrt{\mathfrak{f}^3_{1,1}\mathfrak{f}_{2,2}\mathfrak{f}_{3,3}\mathfrak{f}^3_{4,4}}} \\
&\quad - \frac{1}{2}\frac{\mathfrak{f}_{1,2}\operatorname{Cov}(\mathbb{H}_{2,2},\mathbb{H}_{3,4})}{\sqrt{\mathfrak{f}_{1,1}\mathfrak{f}^3_{2,2}\mathfrak{f}_{3,3}\mathfrak{f}_{4,4}}} + \frac{1}{4}\frac{\mathfrak{f}_{1,2}\overline{\mathfrak{f}_{3,4}}\operatorname{Cov}(\mathbb{H}_{2,2},\mathbb{H}_{3,3})}{\sqrt{\mathfrak{f}_{1,1}\mathfrak{f}^3_{2,2}\mathfrak{f}^3_{3,3}\mathfrak{f}_{4,4}}} + \frac{1}{4}\frac{\mathfrak{f}_{1,2}\overline{\mathfrak{f}_{3,4}}\operatorname{Cov}(\mathbb{H}_{2,2},\mathbb{H}_{4,4})}{\sqrt{\mathfrak{f}_{1,1}\mathfrak{f}^3_{2,2}\mathfrak{f}_{3,3}\mathfrak{f}^3_{4,4}}},
\end{aligned}$$

where we have written $\mathfrak{f}_{a,b}$ for the quantile spectral density $\mathfrak{f}^{j_a,j_b}(\omega_{kn};\tau_a,\tau_b)$, and $\mathbb{H}_{a,b}$ for the limit distribution $\mathbb{H}^{j_a,j_b}(\omega_{kn};\tau_a,\tau_b)$ for any $a,b=1,2,3,4$).

Thus, considering the special case where $\tau_3 = \tau_1$ and $\tau_4 = \tau_2$, we have

$$
\begin{aligned}
&\mathrm{Cov}(\mathbb{L}_{1,2}, \mathbb{L}_{1,2}) \\
&= \frac{1}{\mathfrak{f}_{1,1}\mathfrak{f}_{2,2}}\Big(\mathrm{Cov}(\mathbb{H}_{1,2}, \mathbb{H}_{1,2}) - \Re\frac{\mathfrak{f}_{1,2}\,\mathrm{Cov}(\mathbb{H}_{1,1}, \mathbb{H}_{1,2})}{\mathfrak{f}_{1,1}} - \Re\frac{\mathfrak{f}_{1,2}\,\mathrm{Cov}(\mathbb{H}_{2,2}, \mathbb{H}_{1,2})}{\mathfrak{f}_{2,2}} \\
&\quad + \frac{1}{4}|\mathfrak{f}_{1,2}|^2\Big(\frac{\mathrm{Cov}(\mathbb{H}_{1,1}, \mathbb{H}_{1,1})}{\mathfrak{f}_{1,1}^2} + 2\Re\frac{\mathrm{Cov}(\mathbb{H}_{1,1}, \mathbb{H}_{2,2})}{\mathfrak{f}_{1,1}\mathfrak{f}_{2,2}} + \frac{\mathrm{Cov}(\mathbb{H}_{2,2}, \mathbb{H}_{2,2})}{\mathfrak{f}_{2,2}^2}\Big)\Big)
\end{aligned}
\tag{S.12}
$$

and for the special case where $\tau_3 = \tau_1$ and $\tau_4 = \tau_2$ we have

$$
\begin{aligned}
&\mathrm{Cov}(\mathbb{L}_{1,2}, \mathbb{L}_{2,1}) \\
&= \frac{1}{\mathfrak{f}_{1,1}\mathfrak{f}_{2,2}}\Big(\mathrm{Cov}(\mathbb{H}_{1,2}, \mathbb{H}_{2,1}) - \frac{\mathfrak{f}_{1,2}\,\mathrm{Cov}(\mathbb{H}_{1,2}, \mathbb{H}_{2,2})}{\mathfrak{f}_{2,2}} - \frac{\mathfrak{f}_{1,2}\,\mathrm{Cov}(\mathbb{H}_{1,2}, \mathbb{H}_{1,1})}{\mathfrak{f}_{1,1}} \\
&\quad + \frac{1}{4}\mathfrak{f}_{1,2}^2\Big(\frac{\mathrm{Cov}(\mathbb{H}_{1,1}, \mathbb{H}_{1,1})}{\mathfrak{f}_{1,1}^2} + 2\Re\frac{\mathrm{Cov}(\mathbb{H}_{1,1}, \mathbb{H}_{2,2})}{\mathfrak{f}_{1,1}\mathfrak{f}_{2,2}} + \frac{\mathrm{Cov}(\mathbb{H}_{2,2}, \mathbb{H}_{2,2})}{\mathfrak{f}_{2,2}^2}\Big)\Big).
\end{aligned}
$$

We substitute consistent estimators for the unknown quantities. To do so we abuse notation using $\mathfrak{f}_{a,b}$ to denote $\tilde{G}^{j_a,j_b}_{n,R}(\omega_{kn}; \tau_a, \tau_b)$ and write $\mathrm{Cov}(\mathbb{H}_{a,b}, \mathbb{H}_{c,d})$ for the quantity defined in (S.10).

S6. PROOFS OF THE RESULTS IN SECTIONS 4 AND S4

In this section the proofs to the results in Sections **??** and S4 are given. Before we begin, note that by a trivial generalisation of Proposition 3.1 in Kley et al. (2016) we have that Assumption **??** implies that there exist constants $\rho \in (0, 1)$ and $K < \infty$ such that, for arbitrary intervals $A_1, ..., A_p \subset \mathbb{R}$, arbitrary indices $j_1, \dots, j_p \in \{1, \dots, d\}$ and times $t_1, ..., t_p \in \mathbb{Z}$,

$$
|\,\mathrm{cum}(I\{X_{t_1,j_1} \in A_1\}, \dots, I\{X_{t_p,j_p} \in A_p\})| \le K\rho^{\max_{i,j}|t_i - t_j|}. \tag{S.13}
$$

We will use this fact several times throughout the proofs in this section.

S6.1. Proof of Theorem ??

By a Taylor expansion we have, for every $x, x_0 > 0$,

$$
\frac{1}{\sqrt{x}} = \frac{1}{\sqrt{x_0}} - \frac{1}{2}\frac{1}{\sqrt{x_0^3}}(x - x_0) + \frac{3}{8}\xi_{x,x_0}^{-5/2}(x - x_0)^2,
$$

where ξ_{x,x_0} is between x and x_0. Let $R_n(x, x_0) := \frac{3}{8}\xi_{x,x_0}^{-5/2}(x - x_0)^2$, then

$$
\frac{x}{\sqrt{yz}} - \frac{x_0}{\sqrt{y_0 z_0}} = \frac{1}{\sqrt{y_0 z_0}}\Big((x - x_0) - \frac{1}{2}\frac{x_0}{y_0}(y - y_0) - \frac{1}{2}\frac{x_0}{z_0}(z - z_0)\Big) + r_n, \tag{S.14}
$$

where

$$
\begin{aligned}
r_n &= (x - x_0)\Big(- \frac{1}{2}\frac{1}{y_0}(y - y_0) - \frac{1}{2}\frac{1}{z_0}(z - z_0)\Big) \\
&\quad + x\Big(R_n(y, y_0)\sqrt{y_0}\big(1 - \frac{1}{2}\frac{1}{z_0}(z - z_0)\big) + R_n(z, z_0)\sqrt{z_0}\big(1 - \frac{1}{2}\frac{1}{y_0}(y - y_0)\big) \\
&\quad + \frac{1}{4}\frac{1}{y_0}(y - y_0)\frac{1}{z_0}(z - z_0) + \sqrt{y_0 z_0}R_n(y, y_0)R_n(z, z_0)\Big)
\end{aligned}
$$

Write $\mathfrak{f}_{a,b}$ for $\mathfrak{f}^{j_a,j_b}(\omega;\tau_a,\tau_b)$, $\mathfrak{G}_{a,b}$ for $\hat{G}^{j_a,j_b}_{n,R}(\omega;\tau_a,\tau_b)$, and $\boldsymbol{B}_{a,b}$ for $\{\boldsymbol{B}^{(k)}_n(\omega;\tau_a,\tau_b)\}_{j_a,j_b}$ $(a,b=1,2,3,4)$. We want to employ (S.14) and to this end let

$$x := \mathfrak{G}_{a,b} \qquad y := \mathfrak{G}_{a,a} \qquad z := \mathfrak{G}_{b,b}$$
$$x_0 := \mathfrak{f}_{a,b} + \boldsymbol{B}_{a,b} \qquad y_0 := \mathfrak{f}_{a,a} + \boldsymbol{B}_{a,a} \qquad z_0 := \mathfrak{f}_{b,b} + \boldsymbol{B}_{b,b}$$

By Theorem S4.1 the differences $x-x_0$, $y-y_0$, and $z-z_0$ are in $O_p((nb_n)^{-1/2})$, uniformly with respect to τ_1,τ_2. Under the assumption that $nb_n\to\infty$, as $n\to\infty$, this entails $\mathfrak{G}_{a,a}-\boldsymbol{B}_{a,a}\to\mathfrak{f}_{a,a}$, in probability. For $\varepsilon\le\tau_1,\tau_2\le 1-\varepsilon$, we have $\mathfrak{f}_{a,a}>0$, such that, by the Continuous Mapping Theorem we have $(\mathfrak{G}_{a,a}-\boldsymbol{B}_{a,a})^{-5/2}\to\mathfrak{f}_{a,a}^{-5/2}$, in probability. As $\boldsymbol{B}_{a,a}=o(1)$, we have $y^{-5/2}-y_0^{-5/2}=o_p(1)$. Finally, due to

$$\xi_{y,y_0}^{-5/2}\le y_n^{-5/2}\vee y_0^{-5/2}\le (y_n^{-5/2}-y_0^{-5/2})\vee 0 + y_0^{-5/2} = o_p(1)+O(1)=O_p(1),$$

we have that $R_n(y,y_0)=O_p((nb_n)^{-1})$.
Analogous arguments yields $R_n(z,z_0)=O_p((nb_n)^{-1})$. Thus we have shown that

$$\begin{aligned}&\hat{\mathfrak{R}}^{j_1,j_2}_{n,R}(\omega;\tau_1,\tau_2)-\frac{\mathfrak{f}_{a,b}+\boldsymbol{B}_{a,b}}{\sqrt{\mathfrak{f}_{a,a}+\boldsymbol{B}_{a,a}}\sqrt{\mathfrak{f}_{b,b}+\boldsymbol{B}_{b,b}}}\\ &=\frac{1}{\sqrt{\mathfrak{f}_{1,1}\mathfrak{f}_{2,2}}}\Big([\mathfrak{G}_{1,2}-\mathfrak{f}_{1,2}-\boldsymbol{B}_{1,2}]-\frac{1}{2}\frac{\mathfrak{f}_{1,2}}{\mathfrak{f}_{1,1}}[\mathfrak{G}_{1,1}-\mathfrak{f}_{1,1}-\boldsymbol{B}_{1,1}]-\frac{1}{2}\frac{\mathfrak{f}_{1,2}}{\mathfrak{f}_{2,2}}[\mathfrak{G}_{2,2}-\mathfrak{f}_{2,2}-\boldsymbol{B}_{2,2}]\Big)\\ &\qquad + O_p\big(1/(nb_n)\big),\end{aligned}$$

with the O_p holding uniformly with respect to τ_1,τ_2. Further more, note that

$$\begin{aligned}\frac{\mathfrak{f}_{a,b}+\boldsymbol{B}_{a,b}}{\sqrt{\mathfrak{f}_{a,a}+\boldsymbol{B}_{a,a}}\sqrt{\mathfrak{f}_{b,b}+\boldsymbol{B}_{b,b}}}&=\frac{\mathfrak{f}_{a,b}}{\sqrt{\mathfrak{f}_{a,a}\mathfrak{f}_{b,b}}}+\frac{1}{\sqrt{\mathfrak{f}_{a,a}\mathfrak{f}_{b,b}}}\Big(\boldsymbol{B}_{a,b}-\frac{1}{2}\frac{\mathfrak{f}_{a,b}}{\mathfrak{f}_{a,a}}\boldsymbol{B}_{a,a}-\frac{1}{2}\frac{\mathfrak{f}_{a,b}}{\mathfrak{f}_{b,b}}\boldsymbol{B}_{b,b}\Big)\\ &\quad + O(|\boldsymbol{B}_{a,b}|(\boldsymbol{B}_{a,a}+\boldsymbol{B}_{b,b})+\boldsymbol{B}_{a,a}^2+\boldsymbol{B}_{b,b}^2+\boldsymbol{B}_{a,a}\boldsymbol{B}_{b,b}),\end{aligned}$$

where we have used (S.14) again. By a trivial, multivariate extension of Lemma A.3 in Kley et al. (2016) we have that

$$\sup_{\tau_1,\tau_2\in[\varepsilon,1-\varepsilon]}\Big|\frac{\mathrm{d}^\ell}{\mathrm{d}\omega^\ell}\mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2)\Big|\le C_{\varepsilon,\ell}.$$

Therefore, b_n satisfies

$$\sup_{\tau_1,\tau_2\in[\varepsilon,1-\varepsilon]}\Big|\sum_{\ell=2}^{k}\frac{b_n^\ell}{\ell!}\int_{-\pi}^{\pi}v^\ell W(v)dv\frac{\mathrm{d}^\ell}{\mathrm{d}\omega^\ell}\mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2)\Big|=o\big((nb_n)^{-1/4}\big),$$

for all $j_1,j_2=1,\ldots,d$, which implies that

$$|\boldsymbol{B}_{a,b}|(\boldsymbol{B}_{a,a}+\boldsymbol{B}_{b,b})+\boldsymbol{B}_{a,a}^2+\boldsymbol{B}_{b,b}^2+\boldsymbol{B}_{a,a}\boldsymbol{B}_{b,b}=o\big((nb_n)^{-1/2}\big).$$

Therefore,

$$\begin{aligned}\sqrt{nb_n}\Big(&\hat{\mathfrak{R}}^{j_1,j_2}_{n,R}(\omega;\tau_1,\tau_2)-\mathfrak{R}^{j_1,j_2}(\omega;\tau_1,\tau_2)\\ &-\frac{1}{\sqrt{\mathfrak{f}_{a,a}\mathfrak{f}_{b,b}}}\Big(\boldsymbol{B}_{a,b}-\frac{1}{2}\frac{\mathfrak{f}_{a,b}}{\mathfrak{f}_{a,a}}\boldsymbol{B}_{a,a}-\frac{1}{2}\frac{\mathfrak{f}_{a,b}}{\mathfrak{f}_{b,b}}\boldsymbol{B}_{b,b}\Big)\Big)_{\tau_1,\tau_2\in[0,1]}\end{aligned}$$

and

$$\frac{\sqrt{nb_n}}{\sqrt{\mathfrak{f}_{1,1}\mathfrak{f}_{2,2}}}\Big([\mathfrak{G}_{1,2}-\mathfrak{f}_{1,2}-\boldsymbol{B}_{1,2}]-\frac{1}{2}\frac{\mathfrak{f}_{1,2}}{\mathfrak{f}_{1,1}}[\mathfrak{G}_{1,1}-\mathfrak{f}_{1,1}-\boldsymbol{B}_{1,1}]-\frac{1}{2}\frac{\mathfrak{f}_{1,2}}{\mathfrak{f}_{2,2}}[\mathfrak{G}_{2,2}-\mathfrak{f}_{2,2}-\boldsymbol{B}_{2,2}]\Big) \quad \text{(S.15)}$$

are asymptotically equivalent in the sense that if one of the two converges weakly in $\ell^\infty_{\mathbb{C}^{d\times d}}([0,1]^2)$, then so does the other. The assertion then follows by Theorem S4.1, Slutzky's lemma and the Continuous Mapping Theorem. □

S6.2. Proof of Proposition S4.1

The proof resembles the proof of Proposition 3.4 in Kley et al. (2016), where the univariate case was handled. For $j = 1,\dots,d$ we have, from the continuity of F_j that the ranks of the random variables $X_{0,j}, ..., X_{n-1,j}$ and $F_j(X_{0,j}), ..., F_j(X_{n-1,j})$ coincide almost surely. Thus, without loss of generality, we can assume that the CCR-periodogram is computed from the unobservable data $(F_j(X_{0,j}))_{j=1,\dots,d}, ..., (F_j(X_{n-1,j}))_{j=1,\dots,d}$. In particular, we can assume the marginals to be uniform.

Applying the Continuous Mapping Theorem afterward, it suffices to prove

$$\Big(n^{-1/2} d^j_{n,R}(\omega;\tau)\Big)_{\tau\in[0,1],j=1,\dots,d} \Rightarrow \Big(\mathbb{D}^j(\omega;\tau)\Big)_{\tau\in[0,1],j=1,\dots,d} \quad \text{in } \ell^\infty_{\mathbb{C}^d}([0,1]), \quad \text{(S.16)}$$

where $\ell^\infty_{\mathbb{C}^d}([0,1])$ is the space of bounded functions $[0,1]\to\mathbb{C}^d$ that we identify with the product space $\ell^\infty([0,1])^{2d}$. Let

$$d^j_{n,U}(\omega;\tau) := \sum_{t=0}^{n-1} I\{F_j(X_{t,j}) \le \tau\}\mathrm{e}^{-\mathrm{i}\omega t},$$

$j = 1,\dots,d$, $\omega\in\mathbb{R}$, $\tau\in[0,1]$, and note that for (S.16) to hold, it is sufficient that

$$\big(n^{-1/2} d^j_{n,U}(\omega;\tau)\big)_{\tau\in[0,1],j=1,\dots,d}$$

satisfies the following two conditions:

(i1) convergence of the finite-dimensional distributions, i. e.,

$$\big(n^{-1/2} d^{j_\ell}_{n,U}(\omega_\ell;\tau_\ell)\big)_{\ell=1,\dots,k} \xrightarrow{d} \big(\mathbb{D}^{j_\ell}(\omega_\ell;\tau_\ell)\big)_{\ell=1,\dots,k}, \quad \text{(S.17)}$$

for any $(j_\ell,\tau_\ell)\in\{1,\dots,d\}\times[0,1]$, $\omega_\ell\neq 0 \mod 2\pi$, $\ell=1,\dots,k$ and $k\in\mathbb{N}$;

(i2) stochastic equicontinuity: for any $x>0$ and any $\omega\neq 0 \mod 2\pi$,

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\mathbb{P}\Big(\sup_{\substack{\tau_1,\tau_2\in[0,1]\\|\tau_1-\tau_2|\le\delta}} |n^{-1/2}(d^j_{n,U}(\omega;\tau_1)-d^j_{n,U}(\omega;\tau_2))| > x\Big) = 0, \quad \forall j=1,\dots,d. \quad \text{(S.18)}$$

Under (i1) and (i2), an application of Theorems 1.5.4 and 1.5.7 from van der Vaart and Wellner (1996) then yields

$$\Big(n^{-1/2} d^j_{n,U}(\omega;\tau)\Big)_{\tau\in[0,1],j=1,\dots,d} \Rightarrow \Big(\mathbb{D}^j(\omega;\tau)\Big)_{\tau\in[0,1],j=1,\dots,d} \quad \text{in } \ell^\infty_{\mathbb{C}^d}([0,1]). \quad \text{(S.19)}$$

In combination with

$$\sup_{\tau\in[0,1]} |n^{-1/2}(d^j_{n,R}(\omega;\tau) - d^j_{n,U}(\omega;\tau))| = o_p(1), \quad \text{for } \omega \neq 0 \mod 2\pi,\ j = 1,\dots,d, \tag{S.20}$$

which we will prove below, (S.19) yields the desired result: (S.16). For the proof of (S.20), we denote by $\hat{F}^{-1}_{n,j}(\tau) := \inf\{x : \hat{F}_{n,j}(x) \geq \tau\}$ the generalised inverse of $\hat{F}_{n,j}$ and let $\inf\emptyset := 0$. Then, we have, as in (7.25) of Kley et al. (2016), that

$$\sup_{\omega\in\mathbb{R}} \sup_{\tau\in[0,1]} \Big| d^j_{n,R}(\omega;\tau) - d^j_{n,U}(\omega;\hat{F}^{-1}_{n,j}(\tau)) \Big| \leq n \sup_{\tau\in[0,1]} |\hat{F}_{n,j}(\tau) - \hat{F}_{n,j}(\tau-)| = O_p(n^{1/2k}) \tag{S.21}$$

where $\hat{F}_{n,j}(\tau-) := \lim_{\xi\uparrow 0} \hat{F}_{n,j}(\tau-\xi)$. The O_p-bound in (S.21) follows from Lemma S6.7. Therefore, it suffices to bound the terms

$$\sup_{\tau\in[0,1]} n^{-1/2}|d^j_{n,U}(\omega;\hat{F}^{-1}_{n,j}(\tau)) - d^j_{n,U}(\omega,\tau))|, \text{ for all } j = 1,\dots,d.$$

To do so, note that, for any $x > 0$ and $\delta_n = o(1)$ satisfying $n^{1/2}\delta_n \to \infty$, we have

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{\tau\in[0,1]} n^{-1/2}|d^j_{n,U}(\omega;\hat{F}^{-1}_{n,j}(\tau)) - d^j_{n,U}(\omega;\tau))| > x\Big) \\
&\leq \mathbb{P}\Big(\sup_{\tau\in[0,1]} \sup_{|u-\tau|\leq\delta_n} |d^j_{n,U}(\omega;u) - d^j_{n,U}(\omega;\tau)| > xn^{1/2}, \sup_{\tau\in[0,1]} |\hat{F}^{-1}_{n,j}(\tau) - \tau| \leq \delta_n \Big) \\
&\quad + \mathbb{P}\Big(\sup_{\tau\in[0,1]} |\hat{F}^{-1}_{n,j}(\tau) - \tau| > \delta_n \Big) = o(1) + o(1).
\end{aligned}$$

The first $o(1)$ follows from (S.18). The second one is a consequence of Lemma S6.8.

It thus remains to prove (S.17) and (S.18). For any fixed $j = 1,\dots,d$ the process $\big(d^j_{n,U}(\omega,\tau)\big)_{\tau\in[0,1]}$ is determined by the univariate time series $X_{0,j},\dots,X_{n-1,j}$. Under the assumptions made here, (S.18) therefore follows from (8.7) in Kley et al. (2016).

Finally, we establish (S.17), by employing Lemma S6.6 in combination with Lemma P4.5 and Theorem 4.3.2 from Brillinger (1975). More precisely, to apply Lemma P4.5 from Brillinger (1975), we have to verify that, for any $j_1,\dots,j_\ell \in \{1,\dots,d\}$, $\tau_1,\dots,\tau_\ell \in [0,1]$, $\ell \in \mathbb{N}$, and $\omega_1,\dots,\omega_\ell \neq 0 \mod 2\pi$, all cumulants of the vector

$$n^{-1/2}\big(d^{j_1}_{n,U}(\omega_1;\tau_1), d^{j_1}_{n,U}(-\omega_1;\tau_1),\dots,d^{j_\ell}_{n,U}(\omega_\ell;\tau_\ell), d^{j_\ell}_{n,U}(-\omega_\ell;\tau_\ell)\big)$$

converge to the corresponding cumulants of the vector

$$\big(\mathbb{D}^{j_1}(\omega_1;\tau_1), \mathbb{D}^{j_1}(-\omega_1;\tau_1),\dots,\mathbb{D}^{j_\ell}(\omega_\ell;\tau_\ell), \mathbb{D}^{j_\ell}(-\omega_\ell;\tau_\ell)\big).$$

For the cumulants of order one the arguments from the univariate case (cf. the proof of Proposition 3.4 in Kley et al. (2016)) apply: we have $|E(n^{-1/2}d^j_{n,U}(\omega;\tau))| = o(1)$, for any $j = 1,\dots,d$, $\tau \in [0,1]$ and fixed $\omega \neq 0 \mod 2\pi$. Furthermore, for the cumulants of order two, applying Theorem 4.3.1 in Brillinger (1975) to the bivariate process

$$(I\{X_{t,j_1} \leq q_{j_1}(\mu_1)\}, I\{X_{t,j_2} \leq q_{j_2}(\mu_2)\}),$$

we obtain

$$\mathrm{cum}(n^{-1/2}d^{i_1}_{n,U}(\lambda_1;\mu_1), n^{-1/2}d^{i_2}_{n,U}(\lambda_2;\mu_2)) = 2\pi n^{-1}\Delta_n(\lambda_1+\lambda_2)\mathfrak{f}^{i_1,i_2}(\lambda_1;\mu_1,\mu_2) + o(1)$$

for any $(i_1,\lambda_1,\mu_1),(i_2,\lambda_2,\mu_2) \in \bigcup_{\ell=1}^k \{(i_\ell,\omega_\ell,\tau_\ell),(j_\ell,-\omega_\ell,\tau_\ell)\}$, which yields the correct second moment structure. The function Δ_n is defined in Lemma S6.6. Finally, the cumulants of order J, with $J \in \mathbb{N}$ and $J \geq 3$, all tend to zero, as in view of Lemma S6.6

$$\begin{aligned}\operatorname{cum}(n^{-1/2} d^{i_1}_{n,U}(\lambda_1;\mu_1),\ldots,n^{-1/2} d^{i_J}_{n,U}(\lambda_J;\mu_J))\\ \leq C n^{-J/2}(|\Delta_n(\sum_{j=1}^{J}\lambda_j)|+1)\varepsilon(|\log\varepsilon|+1)^d = O(n^{-(J-2)/2}) = o(1),\end{aligned}$$

for $(i_1,\lambda_1,\mu_1),\ldots,(i_J,\lambda_J,\mu_J) \in \bigcup_{\ell=1}^{k}\{(i_\ell,\omega_\ell,\tau_\ell),(i_\ell,-\omega_\ell,\tau_\ell)\}$, where $\varepsilon := \min_{j=1}^{J}\mu_j$. This implies that the limit $\mathbb{D}^j(\tau;\omega)$ is Gaussian, and completes the proof of (S.17). Proposition S4.1 follows. □

S6.3. Proof of Theorem S4.1

We proceed in a similar fashion as in the proof of the univariate estimator which was analysed in Kley et al. (2016). First, we state an asymptotic representation result by which the estimator $\hat{\boldsymbol{G}}_{n,R}$ can be approximated, in a suitable uniform sense, by another process $\hat{\boldsymbol{G}}_{n,U}$ which is not defined as a function of the standardised ranks $\hat{F}_{n,j}(X_{t,j})$, but as a function of the unobservable quantities $F_j(X_{t,j})$, $t = 0,\ldots,n-1$, $j = 1,\ldots,d$. More precisely, this process is defined as

$$\hat{\boldsymbol{G}}_{n,U}(\omega;\tau_1,\tau_2) := (\hat{G}^{j_1,j_2}_{n,U}(\omega;\tau_1,\tau_2))_{j_1,j_2=1,\ldots,d},$$

where

$$\begin{aligned}\hat{G}^{j_1,j_2}_{n,U}(\omega;\tau_1,\tau_2) &:= \frac{2\pi}{n}\sum_{s=1}^{n-1} W_n\big(\omega-2\pi s/n\big) I^{j_1,j_2}_{n,U}(2\pi s/n,\tau_1,\tau_2)\\ I^{j_1,j_2}_{n,U}(\omega;\tau_1,\tau_2) &:= \frac{1}{2\pi n} d^{j_1}_{n,U}(\omega;\tau_1) d^{j_2}_{n,U}(-\omega;\tau_2)\\ d^{j}_{n,U}(\omega;\tau) &:= \sum_{t=0}^{n-1} I\{F_j(X_{t,j}) \leq \tau\}\mathrm{e}^{-\mathrm{i}\omega t}. \qquad \text{(S.22)}\end{aligned}$$

Theorem S4.1 then follows from the asymptotic representation of $\hat{\boldsymbol{G}}_{n,R}$ by $\hat{\boldsymbol{G}}_{n,U}$ (i. e., Theorem S6.1(iii)) and the asymptotic properties of $\hat{\boldsymbol{G}}_{n,U}$ (i. e., Theorem S6.1(i)–(ii)), which we now state:

Theorem S6.1. *Let Condition* (S.13) *and Assumption* **??** *hold, and assume that the distribution functions F_j of $X_{0,j}$ are continuous for all $j = 1,\ldots,d$. Let b_n satisfy the assumptions of Theorem S4.1. Then,*

(i) for any fixed $\omega \in \mathbb{R}$, as $n \to \infty$,

$$\sqrt{nb_n}\big(\hat{\boldsymbol{G}}_{n,U}(\omega;\tau_1,\tau_2) - E\hat{\boldsymbol{G}}_{n,U}(\omega;\tau_1,\tau_2)\big)_{\tau_1,\tau_2\in[0,1]} \Rightarrow \mathbb{H}(\omega;\cdot,\cdot)$$

in $\ell^\infty_{\mathbb{C}^{d\times d}}([0,1]^2)$, where the process $\mathbb{H}(\omega;\cdot,\cdot)$ is defined in Theorem S4.1;

(ii) still as $n \to \infty$,

$$\sup_{\substack{j_1,j_2\in\{1,\dots,d\}\\ \tau_1,\tau_2\in[0,1]\\ \omega\in\mathbb{R}}} \Big| E\hat{G}^{j_1,j_2}_{n,U}(\tau_1,\tau_2;\omega) - \mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2) - \big\{\boldsymbol{B}^{(k)}_n(\omega;\tau_1,\tau_2)\big\}_{j_1,j_2}\Big|$$

$$= O((nb_n)^{-1}) + o(b_n^k),$$

where $\big\{\boldsymbol{B}^{(k)}_n(\omega;\tau_1,\tau_2)\big\}_{j_1,j_2}$ *is defined in* (S.7);

(iii) for any fixed $\omega\in\mathbb{R}$,

$$\sup_{\substack{j_1,j_2\in\{1,\dots,d\}\\ \tau_1,\tau_2\in[0,1]}} |\hat{G}^{j_1,j_2}_{n,R}(\tau_1,\tau_2;\omega) - \hat{G}^{j_1,j_2}_{n,U}(\tau_1,\tau_2;\omega)| = o_p\big((nb_n)^{-1/2} + b_n^k\big);$$

if moreover the kernel W *is uniformly Lipschitz-continuous, this bound is uniform with respect to* $\omega\in\mathbb{R}$.

The proof of Theorem S6.1 is lengthy, technical and in many places similar to the proof of Theorem 3.6 in Kley et al. (2016). We provide the proof in Sections S6.3.1–S6.3.3, with technical details deferred to Section S6.4. For the reader's convenience we first give a brief description of the necessary steps.

Part (ii) of Theorem S6.1 can be proved along the lines of classical results from Brillinger (1975), but uniformly with respect to the arguments τ_1 and τ_2. Parts (i) and (iii) require additional arguments that are different from the classical theory. These additional arguments are due to the fact that the estimator is a stochastic process and stochastic equicontinuity of

$$\big(\hat{H}^{j_1,j_2}_n(a;\omega)\big)_{a\in[0,1]^2} := \sqrt{nb_n}\big(\hat{G}^{j_1,j_2}_{n,U}(\omega;\tau_1,\tau_2) - E\hat{G}^{j_1,j_2}_{n,U}(\omega;\tau_1,\tau_2)\big)_{\tau_1,\tau_2\in[0,1]} \quad \text{(S.23)}$$

for all $j_1,j_2 = 1,\dots,d$ has to be proven to ensure that the convergence holds not only pointwise, but also uniformly. The key to the proof of (i) and (iii) is a uniform bound on the increments $\hat{H}^{j_1,j_2}_n(a;\omega) - \hat{H}^{j_1,j_2}_n(b;\omega)$ of the process $\hat{H}^{j_1,j_2}_n$. This bound is needed to show the stochastic equicontinuity of the process. To employ a restricted chaining technique (cf. Lemma S6.3), we require two different bounds. First, we prove a general bound, uniform in a and b, on the moments of the increments $\hat{H}^{j_1,j_2}_n(a;\omega) - \hat{H}^{j_1,j_2}_n(b;\omega)$ (cf. Lemma S6.4). Second, we prove a sharper bound on the increments $\hat{H}^{j_1,j_2}_n(a;\omega) - \hat{H}^{j_1,j_2}_n(b;\omega)$ when a and b are "sufficiently close" (cf. Lemma S6.10).

Condition (S.28) which we will required for Lemma S6.4 to hold is rather general. In Lemma S6.6 we prove that condition (S.13), which is implied by Assumption **??**, implies (S.28).

S6.3.1. Proof of Theorem S6.1(i) It is sufficient to prove the following two claims:

(i1) convergence of the finite-dimensional distributions of the process (S.23), that is,

$$\big(\hat{H}^{j_{1\ell},j_{2\ell}}_n\big((a_{1\ell},a_{2\ell});\omega_j\big)\big)_{j=1,\dots,k} \xrightarrow{d} \big(\mathbb{H}^{j_{1\ell},j_{2\ell}}\big((a_{1\ell},a_{2\ell});\omega_j\big)\big)_{j=1,\dots,k} \quad \text{(S.24)}$$

for any $(j_{1\ell},j_{2\ell},a_{1\ell},a_{2\ell},\omega_\ell)\in\{1,\dots,d\}\times[0,1]^2\times\mathbb{R}$, $\ell=1,\dots,k$ and $k\in\mathbb{N}$;

(i2) stochastic equicontinuity: for any $x>0$, any $\omega\in\mathbb{R}$, and any $j_1,j_2=1,\dots,d$,

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\mathbb{P}\Big(\sup_{\substack{a,b\in[0,1]^2\\ \|a-b\|_1\le\delta}} |\hat{H}^{j_1,j_2}_n(a;\omega) - \hat{H}^{j_1,j_2}_n(b;\omega)| > x\Big) = 0. \quad \text{(S.25)}$$

By (S.25) we have stochastic equicontinuity of all real parts $\Re\hat{H}_n^{j_1,j_2}(\cdot;\omega)$ and imaginary parts $\Im\hat{H}_n^{j_1,j_2}(\cdot;\omega)$. Therefore, in view of Theorems 1.5.4 and 1.5.7 in van der Vaart and Wellner (1996), we will have proven part (i).

First we prove (i1). For fixed τ_1, τ_2, $\hat{G}_{n,U}^{j_1,j_2}(\omega;\tau_1,\tau_2)$ is the traditional smoothed periodogram estimator of the cross-spectrum of the clipped processes $(I\{F_{j_1}(X_{t,j_1}) \le \tau_1\})_{t\in\mathbb{Z}}$ and $(I\{F_{j_2}(X_{t,j_2}) \le \tau_2\})_{t\in\mathbb{Z}}$ [see Chapter 7.1 in Brillinger (1975)]. Thus, (S.24) follows from Theorem 7.4.4 in Brillinger (1975), by which these estimators are asymptotically jointly Gaussian. The first and second moment structures of the limit are given by Theorem 7.4.1 and Corollary 7.4.3 in Brillinger (1975). The joint convergence (S.24) follows. Note that condition (S.13), which is implied by Assumption **??**, implies the summability condition [i. e., Assumption 2.6.2(ℓ) in Brillinger (1975), for every ℓ] required for the three theorems in Brillinger (1975) to be applied.

Now to the proof of (i2). The Orlicz norm $\|X\|_\Psi = \inf\{C > 0 : E\Psi(|X|/C) \le 1\}$ with $\Psi(x) := x^6$ coincides with the L_6 norm $\|X\|_6 = (E|X|^6)^{1/6}$. Therefore, for any $\kappa > 0$ and sufficiently small $\|a-b\|_1$, we have by Lemma S6.4 and Lemma S6.6 that

$$\|\hat{H}_n^{j_1,j_2}(a;\omega) - \hat{H}_n^{j_1,j_2}(b;\omega)\|_\Psi \le K\Big(\frac{\|a-b\|_1^\kappa}{(nb_n)^2} + \frac{\|a-b\|_1^{2\kappa}}{nb_n} + \|a-b\|_1^{3\kappa}\Big)^{1/6}.$$

Consequently, for all a, b with $\|a-b\|_1$ sufficiently small and $\|a-b\|_1 \ge (nb_n)^{-1/\gamma}$ and all $\gamma \in (0,1)$ such that $\gamma < \kappa$,

$$\|\hat{H}_n^{j_1,j_2}(a;\omega) - \hat{H}_n^{j_1,j_2}(b;\omega)\|_\Psi \le \bar{K}\|a-b\|_1^{\gamma/2}.$$

Note that $\|a-b\|_1 \ge (nb_n)^{-1/\gamma}$ if and only if $d(a,b) := \|a-b\|_1^{\gamma/2} \ge (nb_n)^{-1/2} =: \bar{\eta}_n/2$. The packing number (van der Vaart and Wellner, 1996, p. 98) $D(\varepsilon, d)$ of $([0,1]^2, d)$ satisfies $D(\varepsilon, d) \asymp \varepsilon^{-4/\gamma}$. By Lemma S6.3, we therefore have, for all $x, \delta > 0$ and $\eta \ge \bar{\eta}_n$,

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{\|a-b\|_1 \le \delta^{2/\gamma}} |\hat{H}_n^{j_1,j_2}(a;\omega) - \hat{H}_n^{j_1,j_2}(b;\omega)| > x\Big) \\
&= \mathbb{P}\Big(\sup_{d(a,b)\le\delta} |\hat{H}_n^{j_1,j_2}(a;\omega) - \hat{H}_n^{j_1,j_2}(b;\omega)| > x\Big) \\
&\le \left[\frac{8\tilde{K}}{x}\left(\int_{\bar{\eta}_n/2}^{\eta} \epsilon^{-2/(3\gamma)}\mathrm{d}\epsilon + (\delta + 2\bar{\eta}_n)\eta^{-4/(3\gamma)}\right)\right]^6 \\
&\quad + \mathbb{P}\Big(\sup_{d(a,b)\le\bar{\eta}_n} |\hat{H}_n^{j_1,j_2}(a;\omega) - \hat{H}_n^{j_1,j_2}(b;\omega)| > x/4\Big).
\end{aligned}$$

Now, choosing $2/3 < \gamma < 1$ and letting n tend to infinity, the second term tends to zero by Lemma S6.10, because, by construction, $1/\gamma > 1$ and $d(a,b) \le \bar{\eta}_n$ if and only if $\|a-b\|_1 \le 2^{2/\gamma}(nb_n)^{-1/\gamma}$. All together, this yields

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty} \mathbb{P}\Big(\sup_{d(a,b)\le\delta} |\hat{H}_n(a;\omega) - \hat{H}_n(b;\omega)| > x\Big) \le \left[\frac{8\tilde{K}}{x}\int_0^{\eta} \epsilon^{-2/(3\gamma)}\mathrm{d}\epsilon\right]^6,$$

for every $x, \eta > 0$. The claim then follows, as the integral on the right-hand side may be arbitrarily small by choosing η accordingly. □

S6.3.2. Proof of Theorem S6.1(ii) Following the arguments which were applied in Section 8.1 of Kley et al. (2016) we can derive asymptotic expansions for $E[I_{n,U}^{j_1,j_2}(\omega;\tau_1,\tau_2)]$

and $E[\hat{G}^{j_1,j_2}_{n,U}(\omega;\tau_1,\tau_2)]$. In fact, it is easy to see that the proofs can still be applied when the Laplace cumulants

$$\mathrm{cum}\big(I\{X_{k_1} \le x_1\}, I\{X_{k_2} \le x_2\}, \dots, I\{X_0 \le x_p\}\big)$$

which were considered in Kley et al. (2016) are replaced by their multivariate counterparts

$$\mathrm{cum}\big(I\{X_{k_1,j_1} \le x_1\}, I\{X_{k_2,j_2} \le x_2\}, \dots, I\{X_{0,j_p} \le x_p\}\big).$$

More precisely, we now state Lemma S6.1 and S6.2 (without proof) that are multivariate counterparts to Lemmas 8.4 and 8.5 in Kley et al. (2016), for which we assume

Assumption S6.1. *Let $p \ge 2, \delta > 0$. There exists a non-increasing function $a_p : \mathbb{N} \to \mathbb{R}^+$ such that $\sum_{k\in\mathbb{N}} k^\delta a_p(k) < \infty$ and*

$$\sup_{x_1,\dots,x_p} |\,\mathrm{cum}\big(I\{X_{k_1,j_1} \le x_1\}, I\{X_{k_2,j_2} \le x_2\}, \dots, I\{X_{0,j_p} \le x_p\}\big)| \le a_p\big(\max_j |k_j|\big),$$

for all $j_1, \dots, j_p = 1, \dots, d$.

Note that Assumption S6.1 follows from condition (S.13), which is in turn implied by Assumption **??**, but that it is in fact somewhat weaker. We now state the first of the two lemmas. It is a generalisation of Theorem 5.2.2 in Brillinger (1975).

Lemma S6.1. *Under Assumption S6.1 with $K = 2, \delta > 3$,*

$$EI^{j_1,j_2}_{n,U}(\omega;\tau_1,\tau_2) = \begin{cases} \mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2) + \frac{1}{2\pi n}\Big[\frac{\sin(n\omega/2)}{\sin(\omega/2)}\Big]^2\tau_1\tau_2 + \varepsilon_n^{\tau_1,\tau_2}(\omega) & \omega \ne 0 \mod 2\pi \\ \mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2) + \frac{n}{2\pi}\tau_1\tau_2 + \varepsilon_n^{\tau_1,\tau_2}(\omega) & \omega = 0 \mod 2\pi \end{cases} \tag{S.26}$$

with $\sup_{\tau_1,\tau_2\in[0,1],\omega\in\mathbb{R}} |\varepsilon_n^{\tau_1,\tau_2}(\omega)| = O(1/n)$.

The second of the two lemmas is a generalisation of Theorem 5.6.1 in Brillinger (1975).

Lemma S6.2. *Assume that Assumption S6.1, with $p = 2$ and $\delta > k + 1$, and Assumption **??** hold. Then, with the notation of Theorem S4.1,*

$$\sup_{\tau_1,\tau_2\in[0,1],\omega\in\mathbb{R}} \Big| E\hat{G}^{j_1,j_2}_n(\omega;\tau_1,\tau_2) - \mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2) - \big\{\boldsymbol{B}^{(k)}_n(\omega;\tau_1,\tau_2)\big\}_{j_1,j_2}\Big| = O((nb_n)^{-1}) + o(b_n^k).$$

Because condition (S.13), which is implied by Assumption **??**, implies Assumption S6.1, Lemma S6.2 implies Theorem S6.1(ii). □

S6.3.3. Proof of Theorem S6.1(iii) Using (S.20) and argument similar to the ones in the proof of Lemma S6.10 it follows that

$$\sup_{\omega\in\mathbb{R}} \sup_{\tau_1,\tau_2\in[0,1]} |\hat{G}^{j_1,j_2}_{n,R}(\omega;\tau_1,\tau_2) - \hat{G}^{j_1,j_2}_{n,U}(\omega;\hat{F}^{-1}_{n,j_1}(\tau_1),\hat{F}^{-1}_{n,j_2}(\tau_2))| = o_p(1).$$

It therefore suffices to bound the differences

$$\sup_{\tau_1,\tau_2\in[0,1]} |\hat{G}^{j_1,j_2}_{n,U}(\omega;\tau_1,\tau_2) - \hat{G}^{j_1,j_2}_{n,U}(\omega;\hat{F}^{-1}_{n,j_1}(\tau_1),\hat{F}^{-1}_{n,j_2}(\tau_2))|$$

for $j_1, j_2 = 1, \ldots, d$, pointwise and uniformly in ω.

We first prove the statement for fixed $\omega \in \mathbb{R}$ in full details and will later sketch the additional arguments needed for the proof of the uniform result. For any $x > 0$ and sequence δ_n we have,

$$
\begin{aligned}
&P^n(\omega) := \\
&\mathbb{P}\Big(\sup_{\tau_1,\tau_2\in[0,1]} |\hat{G}^{j_1,j_2}_{n,U}(\omega; \hat{F}^{-1}_{n,j_1}(\tau_1), \hat{F}^{-1}_{n,j_2}(\tau_2)) - \hat{G}^{j_1,j_2}_{n,U}(\omega;\tau_1,\tau_2)| > x((nb_n)^{-1/2} + b_n^k)\Big) \\
&\quad \le \mathbb{P}\Big(\sup_{\tau_1,\tau_2\in[0,1]} \sup_{\substack{\|(u,v)-(\tau_1,\tau_2)\|_\infty \\ \le \sup_{i=1,2;\tau\in[0,1]} |\hat{F}^{-1}_{n,j_i}(\tau)-\tau|}} |\hat{G}^{j_1,j_2}_{n,U}(\omega;u,v) - \hat{G}^{j_1,j_2}_{n,U}(\omega;\tau_1,\tau_2)| \\
&\qquad\qquad > x((nb_n)^{-1/2} + b_n^k)\Big) \\
&\quad \le \mathbb{P}\Big(\sup_{\tau_1,\tau_2\in[0,1]} \sup_{\substack{|u-\tau_1|\le\delta_n \\ |v-\tau_2|\le\delta_n}} |\hat{G}^{j_1,j_2}_{n,U}(\omega;u,v) - \hat{G}^{j_1,j_2}_{n,U}(\omega;\tau_1,\tau_2)| > x((nb_n)^{-1/2} + b_n^k), \\
&\qquad \sup_{i=1,2;\tau\in[0,1]} |\hat{F}^{-1}_{n,j_i}(\tau) - \tau| \le \delta_n \Big) + \sum_{i=1}^{2} \mathbb{P}\Big(\sup_{\tau\in[0,1]} |\hat{F}^{-1}_{n,j_i}(\tau) - \tau| > \delta_n \Big) \\
&\quad = P_1^n + P_2^n, \quad \text{say.}
\end{aligned}
$$

We choose δ_n such that $n^{-1/2} \ll \delta_n = o(n^{-1/2} b_n^{-1/2} (\log n)^{-D})$, where D denotes the constant from Lemma S6.5. It then follows from Lemma S6.8 that P_2^n is $o(1)$. For P_1^n, on the other hand, we have the following bound:

$$
\begin{aligned}
&\mathbb{P}\Big(\sup_{\tau_1,\tau_2\in[0,1]} \sup_{\substack{|u-\tau_1|\le\delta_n \\ |v-\tau_2|\le\delta_n}} |\hat{H}^{j_1,j_2}_{n,U}(\omega;u,v) - \hat{H}^{j_1,j_2}_{n,U}(\omega;\tau_1,\tau_2)| > (1 + (nb_n)^{1/2} b_n^k) x/2 \Big) \\
&\quad + I\Big\{ \sup_{\tau_1,\tau_2\in[0,1]} \sup_{\substack{|u-\tau_1|\le\delta_n \\ |v-\tau_2|\le\delta_n}} |E\hat{G}^{j_1,j_2}_{n,U}(\omega;u,v) - E\hat{G}^{j_1,j_2}_{n,U}(\omega;\tau_1,\tau_2)| > ((nb_n)^{-1/2} + b_n^k) x/2 \Big\}.
\end{aligned}
$$

The first term tends to zero because of (S.25). The indicator vanishes for n large enough, because we have

$$
\begin{aligned}
&\sup_{\tau_1,\tau_2\in[0,1]} \sup_{\substack{|u-\tau_1|\le\delta_n \\ |v-\tau_2|\le\delta_n}} |E\hat{G}^{j_1,j_2}_{n,U}(\omega;u,v) - E\hat{G}^{j_1,j_2}_{n,U}(\omega;\tau_1,\tau_2)| \\
&\le \sup_{\tau_1,\tau_2\in[0,1]} \sup_{\substack{|u-\tau_1|\le\delta_n \\ |v-\tau_2|\le\delta_n}} |E\hat{G}^{j_1,j_2}_{n,U}(\omega;u,v) - \mathfrak{f}^{j_1,j_2}(\omega;u,v) - \big\{B_n^{(k)}(\omega;u,v)\big\}_{j_1,j_2}| \\
&\quad + \sup_{\tau_1,\tau_2\in[0,1]} \sup_{\substack{|u-\tau_1|\le\delta_n \\ |v-\tau_2|\le\delta_n}} |\big\{B_n^{(k)}(\omega;\tau_1,\tau_2)\big\}_{j_1,j_2} + \mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2) - E\hat{G}^{j_1,j_2}_{n,U}(\omega;\tau_1,\tau_2)| \\
&\quad + \sup_{\tau_1,\tau_2\in[0,1]} \sup_{\substack{|u-\tau_1|\le\delta_n \\ |v-\tau_2|\le\delta_n}} |\mathfrak{f}^{j_1,j_2}(\omega;u,v) + \big\{B_n^{(k)}(\omega;u,v)\big\}_{j_1,j_2} \\
&\qquad\qquad - \mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2) - \big\{B_n^{(k)}(\omega;\tau_1,\tau_2)\big\}_{j_1,j_2}| \\
&= o(n^{-1/2} b_n^{-1/2} + b_n^k) + O(\delta_n (1 + |\log \delta_n|)^D),
\end{aligned}
$$

where D is still the constant from Lemma S6.5. To bound the first two terms we have applied part (ii) of Theorem S6.1 and Lemma S6.5 for the third one. Thus, for any fixed ω, we have shown $P^n(\omega) = o(1)$, which is the pointwise version of the claim.

Next, we outline the proof of the uniform (with respect to ω) convergence. For any $y_n > 0$, by similar arguments as above, using the same δ_n, we have

$$\mathbb{P}\Big(\sup_{\omega\in\mathbb{R}}\sup_{\tau_1,\tau_2\in[0,1]}|\hat{G}^{j_1,j_2}_{n,R}(\omega;\tau_1,\tau_2) - \hat{G}^{j_1,j_2}_{n,U}(\omega;\tau_1,\tau_2)| > y_n\Big)$$
$$\le \mathbb{P}\Big(\sup_{\omega\in\mathbb{R}}\sup_{\tau_1,\tau_2\in[0,1]}\sup_{\substack{|u-\tau_1|\le\delta_n\\|v-\tau_2|\le\delta_n}}|\hat{H}^{j_1,j_2}_{n,U}(\omega;u,v) - \hat{H}^{j_1,j_2}_{n,U}(\omega;\tau_1,\tau_2)| > (nb_n)^{1/2}y_n/2\Big)$$
$$+ I\Big\{\sup_{\omega\in\mathbb{R}}\sup_{\tau_1,\tau_2\in[0,1]}\sup_{\substack{|u-\tau_1|\le\delta_n\\|v-\tau_2|\le\delta_n}}|E\hat{G}^{j_1,j_2}_{n,U}(\omega;u,v) - E\hat{G}^{j_1,j_2}_{n,U}(\omega;\tau_1,\tau_2)| > y_n/2\Big\} + o(1).$$

The indicator in the latter expression is $o(1)$ by the same arguments as above [note that Lemma S6.5 and the statement of part (ii) both hold uniformly with respect to $\omega \in \mathbb{R}$]. For the bound of the probability, note that by Lemma S6.9,

$$\sup_{\tau_1,\tau_2}\sup_{k=1,\dots,n}|I^{j_1,j_2}_{n,U}(2\pi k/n;\tau_1,\tau_2)| = O_p(n^{2/K}),\ \text{for any } K > 0.$$

Moreover, by the uniform Lipschitz continuity of W the function W_n is also uniformly Lipschitz continuous with constant of order $O(b_n^{-2})$. Combining those facts with Lemma S6.5 and the assumptions on b_n, we obtain

$$\sup_{\substack{\omega_1,\omega_2\in\mathbb{R}\\|\omega_1-\omega_2|\le n^{-3}}}\sup_{\tau_1,\tau_2\in[0,1]}|\hat{H}^{j_1,j_2}_{n,U}(\omega_1;\tau_1,\tau_2) - \hat{H}^{j_1,j_2}_{n,U}(\omega_2;\tau_1,\tau_2)| = o_p(1).$$

By the periodicity of $\hat{H}^{j_1,j_2}_{n,U}$ (with respect to ω), it suffices to show that

$$\max_{\omega=0,2\pi n^{-3},\dots,2\pi}\sup_{\tau_1,\tau_2\in[0,1]}\sup_{\substack{|u-\tau_1|\le\delta_n\\|v-\tau_2|\le\delta_n}}|\hat{H}^{j_1,j_2}_{n,U}(\omega;u,v) - \hat{H}^{j_1,j_2}_{n,U}(\omega;\tau_1,\tau_2)| = o_p(1).$$

By Lemmas S6.3 and S6.10 there exists a random variable $S(\omega)$ such that

$$\sup_{\tau_1,\tau_2\in[0,1]}\sup_{\substack{|u-\tau_1|\le\delta_n\\|v-\tau_2|\le\delta_n}}|\hat{H}^{j_1,j_2}_{n,U}(\omega;u,v) - \hat{H}^{j_1,j_2}_{n,U}(\omega;\tau_1,\tau_2)| \le |S(\omega)| + R_n(\omega),$$

for any fixed $\omega \in \mathbb{R}$, with $\sup_{\omega\in\mathbb{R}}|R_n(\omega)| = o_p(1)$ and

$$\max_{\omega=0,2\pi n^{-3}\dots,2\pi} E[|S^{2L}(\omega)|] \le K_L^{2L}\left(\int_0^\eta \epsilon^{-4/(2L\gamma)}\mathrm{d}\epsilon + (\delta_n^{\gamma/2} + 2(nb_n)^{-1/2})\eta^{-8/(2L\gamma)}\right)^{2L}$$

for any $0 < \gamma < 1, L \in \mathbb{N}$, $0 < \eta < \delta_n$, and a constant K_L depending on L only. For appropriately chosen L and γ, this latter bound is $o(n^{-3})$. Note that the maximum is with respect to a set of cardinality $O(n^3)$, which completes the proof of part (iii). □

S6.4. Auxiliary Lemmas

In this section we state multivariate versions of the auxiliary lemmas from Section 7.4 in Kley et al. (2016). Note that Lemma S6.3 is unaltered and therefore stated without proof. The remaining lemmas are adapted to the multivariate quantities and proofs or directions on how to adapt the proofs in Kley et al. (2016) are collected in the end of this section.

For the statement of Lemma S6.3, we define the Orlicz norm [see e.g. van der Vaart and Wellner (1996), Chapter 2.2] of a real-valued random variable Z as

$$\|Z\|_\Psi = \inf\Big\{C > 0 : E\Psi\Big(|Z|/C\Big) \le 1\Big\},$$

where $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ may be any non-decreasing, convex function with $\Psi(0) = 0$.

For the statement of Lemmas S6.4, S6.6, and S6.9 we define, for any Borel set A,

$$d_n^j(\omega; A) := \sum_{t=0}^{n-1} I\{X_{t,j} \in A\}\mathrm{e}^{-\mathrm{i}t\omega}. \tag{S.27}$$

LEMMA S6.3. *Let $\{\mathbb{G}_t : t \in T\}$ be a separable stochastic process with $\|\mathbb{G}_s - \mathbb{G}_t\|_\Psi \le Cd(s,t)$ for all s, t with $d(s,t) \ge \bar\eta/2 \ge 0$. Denote by $D(\epsilon, d)$ the packing number of the metric space (T, d). Then, for any $\delta > 0$, $\eta \ge \bar\eta$, there exists a random variable S_1 and a constant $K < \infty$ such that*

$$\sup_{d(s,t)\le\delta} |\mathbb{G}_s - \mathbb{G}_t| \le S_1 + 2 \sup_{d(s,t)\le\bar\eta, t\in\tilde T} |\mathbb{G}_s - \mathbb{G}_t| \quad \textit{and}$$

$$\|S_1\|_\Psi \le K\Big[\int_{\bar\eta/2}^{\eta} \Psi^{-1}\big(D(\epsilon, d)\big)\,d\epsilon + (\delta + 2\bar\eta)\Psi^{-1}\big(D^2(\eta, d)\big)\Big],$$

where the set $\tilde T$ contains at most $D(\bar\eta, d)$ points. In particular, by Markov's inequality [cf. van der Vaart and Wellner (1996), p. 96],

$$\mathbb{P}\Big(|S_1| > x\Big) \le \Big(\Psi\Big(x\big[8K\big(\int_{\bar\eta/2}^{\eta} \Psi^{-1}\big(D(\epsilon, d)\big)\,d\epsilon + (\delta + 2\bar\eta)\Psi^{-1}\big(D^2(\eta, d)\big)\big)\big]^{-1}\Big)\Big)^{-1}.$$

for any $x > 0$.

LEMMA S6.4. *Let $\boldsymbol{X}_0, ..., \boldsymbol{X}_{n-1}$, where $\boldsymbol{X}_t = (X_{t,1}, \dots, X_{t,d})$, be the finite realisation of a strictly stationary process with $X_{0,j} \sim U[0,1]$, $j = 1, \dots, d$. Let Assumption* **??** *hold. For $x = (x_1, x_2)$ let $\hat H_n^{j_1,j_2}(x;\omega) := \sqrt{nb_n}(\hat G_n^{j_1,j_2}(x_1, x_2;\omega) - E[\hat G_n^{j_1,j_2}(x_1, x_2;\omega)])$. Let $d_n^j(\omega; A)$ be defined as in* (S.27). *Assume that, for $p = 1, \dots, P$, there exist a constant C and a function $g : \mathbb{R}^+ \to \mathbb{R}^+$, both independent of $\omega_1, ..., \omega_p \in \mathbb{R}, n$ and $A_1, ..., A_p$, such that*

$$\Big|\operatorname{cum}(d_n^{j_1}(\omega_1; A_1), \dots, d_n^{j_p}(\omega_p; A_p))\Big| \le C\Big(\Big|\Delta_n\Big(\sum_{i=1}^{p}\omega_i\Big)\Big| + 1\Big)g(\varepsilon) \tag{S.28}$$

for any indices $j_1, \dots, j_p \in \{1, \dots, d\}$ and intervals $A_1, \dots, A_p$ with $\min_k \mathbb{P}(X_{0,j_k} \in A_k) \le \varepsilon$. Then, there exists a constant K (depending on C, L, g only) such that

$$\sup_{\omega\in\mathbb{R}} \sup_{\|a-b\|_1\le\varepsilon} E|\hat H_n^{j_1,j_2}(a;\omega) - \hat H_n^{j_1,j_2}(b;\omega)|^{2L} \le K \sum_{\ell=0}^{L-1} \frac{g^{L-\ell}(\varepsilon)}{(nb_n)^\ell}$$

for all ε with $g(\varepsilon) < 1$ and all $L = 1, \dots, P$.

LEMMA S6.5. *Under the assumptions of Theorem S4.1, the derivative*

$$(\tau_1, \tau_2) \mapsto \frac{\mathrm{d}^k}{\mathrm{d}\omega^k}\mathfrak{f}^{j_1,j_2}(\omega; \tau_1, \tau_2)$$

exists and satisfies, for any $k \in \mathbb{N}_0$ and some constants C, d that are independent of $a = (a_1, a_2), b = (b_1, b_2)$, but may depend on k,

$$\sup_{\omega \in \mathbb{R}} \Big| \frac{\mathrm{d}^k}{\mathrm{d}\omega^k} \mathfrak{f}^{j_1,j_2}(\omega; a_1, a_2) - \frac{\mathrm{d}^k}{\mathrm{d}\omega^k} \mathfrak{f}^{j_1,j_2}(\omega; b_1, b_2) \Big| \le C\|a-b\|_1 (1 + |\log \|a-b\|_1|)^D.$$

LEMMA S6.6. *Let the strictly stationary process $(\boldsymbol{X}_t)_{t\in\mathbb{Z}}$ satisfy condition* (S.13). *Let $d_n^j(\omega; A)$ be defined as in* (S.27). *Let $A_1, \ldots, A_p \subset [0,1]$ be intervals, and let*

$$\varepsilon := \min_{k=1,\ldots,p} \mathbb{P}(X_{0,j_k} \in A_k).$$

Then, for any p-tuple $\omega_1, ..., \omega_p \in \mathbb{R}$ and $j_1, \ldots, j_p \in \{1, \ldots, d\}$,

$$\Big| \mathrm{cum}(d_n^{j_1}(\omega_1; A_1), \ldots, d_n^{j_p}(\omega_p; A_p)) \Big| \le C\Big(\Big| \Delta_n \Big(\sum_{i=1}^p \omega_i \Big) \Big| + 1 \Big) \varepsilon (|\log \varepsilon| + 1)^D,$$

where $\Delta_n(\lambda) := \sum_{t=0}^{n-1} \mathrm{e}^{\mathrm{i}t\lambda}$ and the constants C, D depend only on K, p, and ρ [with ρ from condition (S.13)*].*

LEMMA S6.7. *Let the strictly stationary process $(\boldsymbol{X}_t)_{t\in\mathbb{Z}}$ satisfy condition* (S.13) *and $X_{0,j} \sim U[0,1]$. Denote the empirical distribution function of $X_{0,j}, ..., X_{n-1,j}$ by $\hat{F}_{n,j}$. Then, for any $k \in \mathbb{N}$, there exists a constant d_k depending only on k, such that*

$$\sup_{x,y\in[0,1], |x-y|\le\delta_n} \sqrt{n} |\hat{F}_{n,j}(x) - \hat{F}_{n,j}(y) - (x-y)| = O_p\Big((n^2\delta_n + n)^{1/2k} (\delta_n |\log \delta_n|^{d_k} + n^{-1})^{1/2} \Big),$$

as $\delta_n \to 0$.

LEMMA S6.8. *Let $\boldsymbol{X}_0, ..., \boldsymbol{X}_{n-1}$, where $\boldsymbol{X}_t = (X_{t,1}, \ldots, X_{t,d})$, be the finite realisation of a strictly stationary process satisfying condition* (S.13) *and $X_{0,j} \sim U[0,1]$, $j = 1, \ldots, d$. Then,*

$$\sup_{j=1,\ldots,d} \sup_{\tau\in[0,1]} |\hat{F}_{n,j}^{-1}(\tau) - \tau| = O_p(n^{-1/2}).$$

LEMMA S6.9. *Let the strictly stationary process $(\boldsymbol{X}_t)_{t\in\mathbb{Z}}$ satisfy condition* (S.13) *and $X_{0,j} \sim U[0,1]$. Let $d_n^j(\omega; A)$ be defined as in* (S.27). *Then, for any $k \in \mathbb{N}$,*

$$\sup_{j=1,\ldots,d} \sup_{\omega\in\mathcal{F}_n} \sup_{y\in[0,1]} |d_n^j(\omega; [0,y])| = O_p(n^{1/2+1/k}).$$

LEMMA S6.10. *Under the assumptions of Theorem S6.1, let δ_n be a sequence of non-negative real numbers. Assume that there exists $\gamma \in (0,1)$, such that $\delta_n = O((nb_n)^{-1/\gamma})$. Then,*

$$\sup_{j_1,j_2,\in\{1,\ldots,d\}} \sup_{\omega\in\mathbb{R}} \sup_{\substack{u,v\in[0,1]^2 \\ \|u-v\|_1\le\delta_n}} |\hat{H}_n^{j_1,j_2}(u;\omega) - \hat{H}_n^{j_1,j_2}(v;\omega)| = o_p(1).$$

Proof of Lemma S6.3. The lemma is stated unaltered as in Kley et al. (2016). The proof can be found in Section 8.3.1 of the Online Appendix of Kley et al. (2016).

Proof of Lemma S6.4. Along the same lines of the proof of the univariate version (Section 8.3.2 in Kley et al. (2016)) we can proof

$$E|\hat{H}_n^{j_1,j_2}(a;\omega) - \hat{H}_n^{j_1,j_2}(b;\omega)|^{2L} = \sum_{\substack{\{\nu_1,\ldots,\nu_R\} \\ |\nu_j|\geq 2,\ j=1,\ldots,R}} \prod_{r=1}^{R} \mathcal{D}_{a,b}(\nu_r) \tag{S.29}$$

with the summation running over all partitions $\{\nu_1,\ldots,\nu_R\}$ of $\{1,\ldots,2L\}$ such that each set ν_j contains at least two elements, and

$$\mathcal{D}_{a,b}(\xi) := \sum_{\ell_{\xi_1},\ldots,\ell_{\xi_q}\in\{1,2\}} n^{-3q/2} b_n^{q/2} \Big(\prod_{m\in\xi} \sigma_{\ell_m}\Big) \times \sum_{s_{\xi_1},\ldots,s_{\xi_q}=1}^{n-1} \Big(\prod_{m\in\xi} W_n(\omega - 2\pi s_m/n)\Big) \operatorname{cum}(D_{\ell_m,(-1)^{m-1}s_m} : m\in\xi),$$

for any set $\xi := \{\xi_1,\ldots,\xi_q\} \subset \{1,\ldots,2L\}$, $q := |\xi|$, and

$$D_{\ell,s} := d_n^{j_1}(2\pi s/n; M_1(\ell)) d_n^{j_2}(-2\pi s/n; M_2(\ell)), \quad \ell = 1,2, \quad s = 1,\ldots,n-1,$$

with the sets $M_1(1)$, $M_2(2)$, $M_2(1)$, $M_1(2)$ and the signs $\sigma_\ell \in \{-1,1\}$ defined as

$$\begin{aligned} \sigma_1 &:= 2I\{a_1 > b_1\} - 1, & \sigma_2 &:= 2I\{a_2 > b_2\} - 1, \\ M_1(1) &:= (a_1\wedge b_1, a_1 \vee b_1], & M_2(2) &:= (a_2\wedge b_2, a_2\vee b_2], \\ M_2(1) &:= \begin{cases} [0,a_2] & b_2 \geq a_2 \\ [0,b_2] & a_2 > b_2, \end{cases} & M_1(2) &:= \begin{cases} [0,b_1] & b_2\geq a_2 \\ [0,a_1] & a_2 > b_2. \end{cases} \end{aligned} \tag{S.30}$$

Employing assumption (S.28), we can further prove, by following the arguments of the univariate version, that

$$\sup_{\substack{\xi\subset\{1,\ldots,2L\} \\ |\xi|=q}} \sup_{\|a-b\|_1\leq\varepsilon} |\mathcal{D}_{a,b}(\xi)| \leq C(nb_n)^{1-q/2} g(\varepsilon), \quad 2\leq q\leq 2L.$$

The lemma then follows, by observing that

$$\Big|\prod_{r=1}^{R} \mathcal{D}_{a,b}(\nu_r)\Big| \leq C g^R(\varepsilon)(nb_n)^{R-L}$$

for any partition in (S.29) [note that $\sum_{r=1}^R |\nu_r| = 2L$]. □

Proof of Lemma S6.5. Note that

$$\begin{aligned} &\operatorname{cum}(I\{X_{0,j_1} \leq q_{j_1}(a_1)\}, I\{X_{k,j_2}\leq q_{j_2}(a_2)\}) \\ &\quad - \operatorname{cum}(I\{X_{0,j_1}\leq q_{j_1}(b_1)\}, I\{X_{k,j_2}\leq q_{j_2}(b_2)\}) \\ &= \sigma_1 \operatorname{cum}(I\{F_{j_1}(X_{0,j_1}) \in M_1(1)\}, I\{F_{j_2}(X_{k,j_2})\in M_2(1)\}) \\ &\qquad + \sigma_2 \operatorname{cum}(I\{F_{j_1}(X_{0,j_1})\in M_1(2)\}, I\{F_{j_2}(X_{k,j_2})\in M_2(2)\}), \end{aligned}$$

with the sets $M_1(1)$, $M_2(2)$, $M_2(1)$, $M_1(2)$ and the signs $\sigma_\ell \in \{-1,1\}$ defined in (S.30).

From the fact that $\lambda(M_j(j)) \leq \|a-b\|_1$ for $j = 1,2$, we conclude that

$$\Big|\frac{\mathrm{d}^\ell}{\mathrm{d}\omega^\ell}\mathfrak{f}^{j_1,j_2}(\omega;a_1,a_2) - \frac{\mathrm{d}^\ell}{\mathrm{d}\omega^\ell}\mathfrak{f}^{j_1,j_2}(\omega;b_1,b_2)\Big|$$
$$\leq \sum_{k\in\mathbb{Z}} |k|^\ell |\operatorname{cum}(I\{F_{j_1}(X_{0,j_1}) \in M_1(1)\}, I\{F_{j_2}(X_{k,j_2}) \in M_2(1)\})|$$
$$+ \sum_{k\in\mathbb{Z}} |k|^\ell |\operatorname{cum}(I\{F_{j_1}(X_{0,j_1}) \in M_1(2)\}, I\{F_{j_2}(X_{k,j_2}) \in M_2(2)\})|$$
$$\leq 4\sum_{k=0}^\infty k^\ell \Big((K\rho^\ell) \wedge \|a-b\|_1\Big).$$

The assertion then follows by after some algebraic manipulations. □

Proof of Lemma S6.6. Similar to (8.27) in Kley et al. (2016) we have, by the definition of cumulants and strict stationarity,

$$\operatorname{cum}(d_n^{j_1}(\omega_1;A_1),\ldots,d_n^{j_p}(\omega_p;A_p))$$
$$= \sum_{u_2,\ldots,u_p=-n}^{n} \operatorname{cum}(I\{X_{0,j_1}\in A_1\}, I\{X_{u_2,j_2}\in A_2\}\ldots, I\{X_{u_p,j_p}\in A_p\}) \exp\Big(-\mathrm{i}\sum_{j=2}^p \omega_j u_j\Big)$$
$$\times \sum_{t_1=0}^{n-1} \exp\Big(-\mathrm{i}t_1\sum_{j=1}^p \omega_j\Big) I_{\{0\leq t_1+u_2<n\}}\cdots I_{\{0\leq t_1+u_p<n\}}. \quad \text{(S.31)}$$

By Lemma 8.1 in Kley et al. (2016),

$$\Big|\Delta_n(\sum_{j=1}^p \omega_j) - \sum_{t_1=0}^{n-1}\exp\Big(-\mathrm{i}t_1\sum_{j=1}^p\omega_j\Big) I\{0\leq t_1+u_2<n\}\cdots I\{0\leq t_1+u_p<n\}\Big|$$
$$\leq 2\sum_{j=2}^p |u_j|. \quad \text{(S.32)}$$

Following the arguments for the proof of (8.29) in Kley et al. (2016), we further have, for any $p+1$ intervals $A_0,\ldots,A_p \subset \mathbb{R}$, any indices $j_0,\ldots,j_p \in \{1,\ldots,d\}$, and any p-tuple $\kappa := (\kappa_1,...,\kappa_p) \in \mathbb{R}_+^p$, $p \geq 2$, that

$$\sum_{k_1,\ldots,k_p=-\infty}^{\infty} \Big(1+\sum_{\ell=1}^p |k_\ell|^{\kappa_\ell}\Big) \big|\operatorname{cum}\big(I\{X_{k_1,j_1}\in A_1\},\ldots,I\{X_{k_p,j_p}\in A_p\}, I\{X_{0,j_0}\in A_0\}\big)\big|$$
$$\leq C\varepsilon(|\log\varepsilon|+1)^d. \quad \text{(S.33)}$$

To this end, define $k_0 = 0$, consider the set

$$T_m := \big\{(k_1,...,k_p)\in\mathbb{Z}^p \big| \max_{i,j=0,\ldots,p}|k_i-k_j| = m\big\},$$

and note that $|T_m| \leq c_p m^{p-1}$ for some constant c_p. From the definition of cumulants and some simple algebra we get the bound

$$|\operatorname{cum}(I\{X_{t_1,j_1}\in A_1\},...,I\{X_{t_p,j_p}\in A_p\})| \leq C\min_{i=1,\ldots,p} P(X_{0,j_i}\in A_i).$$

With this bound and condition (S.13), which is implied by Assumption **??**, we obtain, employing the above notation, that

$$\sum_{k_1,\dots,k_p=-\infty}^{\infty}\Big(1+\sum_{j=1}^{p}|k_\ell|^{\kappa_\ell}\Big)\big|\operatorname{cum}\big(I\{X_{k_1,j_1}\in A_1\},\dots,I\{X_{k_p,j_p}\in A_p\},I_{\{X_{0,j_0}\in A_0\}}\big)\big|$$

$$=\sum_{m=0}^{\infty}\sum_{(k_1,\dots,k_p)\in T_m}\Big(1+\sum_{\ell=1}^{p}|k_\ell|^{\kappa_\ell}\Big)\big|\operatorname{cum}\big(I\{X_{k_1,j_1}\in A_1\},\dots,I\{X_{k_p,j_p}\in A_p\},I\{X_{0,j_0}\in A_0\}\big)\big|$$

$$\le\sum_{m=0}^{\infty}\sum_{(k_1,\dots,k_p)\in T_m}\Big(1+pm^{\max_j\kappa_j}\Big)\Big(\rho^m\wedge\varepsilon\Big)K_p\le C_p\sum_{m=0}^{\infty}\Big(\rho^m\wedge\varepsilon\Big)|T_m|m^{\max_j\kappa_j}.$$

For $\varepsilon\ge\rho$, (S.33) then follows trivially. For $\varepsilon<\rho$, set $m_\varepsilon:=\log\varepsilon/\log\rho$ and note that $\rho^m\le\varepsilon$ if and only if $m\ge m_\varepsilon$. Thus,

$$\sum_{m=0}^{\infty}\Big(\rho^m\wedge\varepsilon\Big)m^u\le\sum_{m\le m_\varepsilon}m^u\varepsilon+\sum_{m>m_\varepsilon}m^u\rho^m\le C\Big(\varepsilon m_\varepsilon^{u+1}+\rho^{m_\varepsilon}\sum_{m=0}^{\infty}(m+m_\varepsilon)^u\rho^m\Big).$$

The fact that $\rho^{m_\varepsilon}=\varepsilon$ completes the proof of the desired inequality (S.33). The assertion follows from (S.31), (S.32), (S.33) and the triangle inequality. □

Proofs of Lemmas S6.7, S6.8 and S6.9. Note that the component processes $(X_{t,j})$ are stationary and fulfill Assumption (C) in Kley et al. (2016), for every $j=1,\dots,d$. The assertion then follow from the univariate versions (i. e., Lemma 8.6, 7.5 and 7.6 in Kley et al. (2016), respectively), as the dimension d does not depend on n. □

Proof of Lemma S6.10. Assume, without loss of generality, that $n^{-1}=o(\delta_n)$ [otherwise, enlarge the supremum by considering $\tilde\delta_n:=\max(n^{-1},\delta_n)$]. With the notation $a=(a_1,a_2)$ and $b=(b_1,b_2)$, we have

$$\hat H_n^{j_1,j_2}(a;\omega)-\hat H_n^{j_1,j_2}(b;\omega)=b_n^{1/2}n^{-1/2}\sum_{s=1}^{n-1}W_n(\omega-2\pi s/n)(K_{s,n}(u,v)-EK_{s,n}(u,v))$$

where, with $d_{n,U}^{j}$ defined in (S.22),

$$\begin{aligned}K_{s,n}(a,b):=&\,n^{-1}\big(d_{n,U}^{j_1}(2\pi s/n;u_1)d_{n,U}^{j_2}(-2\pi s/n;u_2)-d_{n,U}^{j_1}(2\pi s/n;v_1)d_{n,U}^{j_2}(-2\pi s/n;v_2)\big)\\=&\,d_{n,U}^{j_1}(2\pi s/n;u_1)n^{-1}\big[d_{n,U}^{j_2}(-2\pi s/n;u_2)-d_{n,U}^{j_2}(-2\pi s/n;v_2)\big]\\&+d_{n,U}^{j_2}(-2\pi s/n;v_2)n^{-1}\big[d_{n,U}^{j_1}(2\pi s/n;u_1)-d_{n,U}^{j_1}(2\pi s/n;v_1)\big].\end{aligned}$$

By Lemma S6.9, we have, for any $k\in\mathbb{N}$,

$$\sup_{y\in[0,1]}\sup_{\omega\in\mathcal{F}_n}|d_{n,U}^{j}(\omega;y)|=O_p\Big(n^{1/2+1/k}\Big).\qquad\text{(S.34)}$$

Employing Lemma S6.7, we have, for any $\ell\in\mathbb{N}$ and $j=1,\dots,d$,

$$\begin{aligned}&\sup_{\omega\in\mathbb{R}}\sup_{y\in[0,1]}\sup_{x:|x-y|\le\delta_n}n^{-1}|d_{n,U}^{j}(\omega;x)-d_{n,U}^{j}(\omega;y)|\\&\le\sup_{y\in[0,1]}\sup_{x:|x-y|\le\delta_n}n^{-1}\sum_{t=0}^{n-1}|I\{F_j(X_{t,j})\le x\}-I\{F_j(X_{t,j})\le y\}|\\&\le\sup_{y\in[0,1]}\sup_{x:|x-y|\le\delta_n}|\hat F_{n,j}(x\vee y)-\hat F_{n,j}(x\wedge y)-x\vee y+x\wedge y|+C\delta_n\\&=O_p\big(\rho_n(\delta_n,\ell)+\delta_n\big),\end{aligned}$$

with $\rho_n(\delta_n,\ell) := n^{-1/2}(n^2\delta_n+n)^{1/2\ell}(\delta_n|\log\delta_n|^{D_\ell}+n^{-1})^{1/2}$, $\hat{F}_{n,j}$ denoting the empirical distribution function of $F_j(X_{0,j}),\dots,F_j(X_{n-1,j})$, and d_ℓ being a constant depending only on ℓ. Combining these arguments and observing that

$$\sup_{\omega\in\mathbb{R}}\sum_{s=1}^{n-1}\Big|W_n(\omega-2\pi s/n)\Big| = O(n) \tag{S.35}$$

yields

$$\sup_{\omega\in\mathbb{R}}\sup_{\substack{u,v\in[0,1]^2\\ \|u-v\|_1\le\delta_n}}\Big|\sum_{s=1}^{n-1}W_n(\omega-2\pi s/n)K_{s,n}(u,v)\Big| = O_p\big(n^{3/2+1/k}(\rho(\delta_n,\ell)+\delta_n)\big). \tag{S.36}$$

With $M_i(j)$, $i,j=1,2$, as defined in (S.30), we have

$$\begin{aligned}
&\sup_{\|a-b\|_1\le\delta_n}\sup_{s=1,\dots,n-1}|EK_{s,n}(a,b)|\\
&\quad\le n^{-1}\sup_{\|a-b\|_1\le\delta_n}\sup_{s=1,\dots,n-1}\Big|\operatorname{cum}(d^{j_1}_{n,U}(2\pi s/n;M_1(1)),d^{j_2}_{n,U}(-2\pi s/n;M_2(1)))\Big|\\
&\quad+n^{-1}\sup_{\|a-b\|_1\le\delta_n}\sup_{s=1,\dots,n-1}\Big|\operatorname{cum}(d^{j_1}_{n,U}(2\pi s/n;M_1(2)),d^{j_2}_{n,U}(-2\pi s/n;M_2(2)))\Big|
\end{aligned} \tag{S.37}$$

where we have used $Ed^j_{n,U}(2\pi s/n;M)=0$. Lemma S6.6 and $\lambda(M_j(j))\le\delta_n$, for $j=1,2$ (with λ denoting the Lebesgue measure over $\mathbb{R}$) yield

$$\begin{aligned}
&\sup_{\|a-b\|_1\le\delta_n}\sup_{s=1,\dots,n-1}|\operatorname{cum}(d^{j_1}_n(2\pi s/n;M_1(j)),d^{j_2}_n(-2\pi s/n;M_2(j)))|\\
&\qquad\le C(n+1)\delta_n(1+|\log\delta_n|)^D,
\end{aligned}$$

It follows that the right-hand side in (S.37) is $O(\delta_n|\log\delta_n|^D)$. Therefore, by (S.35), we obtain

$$\sup_{\omega\in\mathbb{R}}\sup_{\|a-b\|_1\le\delta_n}\Big|b_n^{1/2}n^{-1/2}\sum_{s=1}^{n-1}W_n(\omega-2\pi s/n)EK_{s,n}(a,b)\Big| = O\big((nb_n)^{1/2}\delta_n|\log n|^D\big).$$

In view of the assumption that $n^{-1}=o(\delta_n)$, we have $\delta_n=O(n^{1/2}\rho_n(\delta_n,\ell))$, which, in combination with (S.36), yields

$$\begin{aligned}
&\sup_{\omega\in\mathbb{R}}\sup_{\|a-b\|_1\le\delta_n}|\hat{H}^{j_1,j_2}_n(a;\omega)-\hat{H}^{j_1,j_2}_n(b;\omega)|\\
&=O_p\Big((nb_n)^{1/2}[n^{1/2+1/k}(\rho_n(\delta_n,\ell)+\delta_n)+\delta_n|\log\delta_n|^D]\Big)\\
&=O_p\Big((nb_n)^{1/2}n^{1/2+1/k}\rho_n(\delta_n,\ell)\Big)\\
&=O_p\Big((nb_n)^{1/2}n^{1/k+1/\ell}(n^{-1}\vee\delta_n(\log n)^{D_\ell})^{1/2}\Big)=o_p(1).
\end{aligned}$$

The $o_p(1)$ holds, as we have, for arbitrary k and ℓ,

$$O((nb_n)^{1/2}n^{1/k+1/\ell}\delta_n^{1/2}(\log n)^{D_\ell/2}) = O((nb_n)^{1/2-1/2\gamma}n^{1/k+1/\ell}(\log n)^{D_\ell/2}).$$

The assumptions on b_n imply $(nb_n)^{1/2-1/2\gamma}=o(n^{-\kappa})$ for some $\kappa>0$, such that this latter quantity is $o(1)$ for k,ℓ sufficiently large. The term $(nb_n)^{1/2}n^{1/k+1/\ell}n^{-1/2}$ is handled in a similar fashion. This concludes the proof. □

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